Power Plane Decoupling

Stephen Blake
stebla@blueyonder.co.uk

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Abstract

The problem of how to choose and place the decoupling capacitors for a circuit board is shrouded in engineering folklore. This text is a reaction to this situation; we study the problem by solving Maxwell's equations and derive guidelines for power-plane decoupling based on the field solutions.

1 Power Plane Impedance

Noise in digital systems is caused by transient currents in the totem-pole outputs of gates when they switch. Figure 1 shows the equivalent circuit; the gate is modeled by a current source I_0 and the power-ground plane pair of the circuit board with its decoupling capacitors is modeled by the impedance Z. The noise voltage across the power/ground plane pair is $V = -I_0 Z$: in order to reduce power-plane noise, the designer should try to minimize the power-plane impedance Z across the whole frequency spectrum. In order to be able to minimize the power-plane impedance, we need some means of calculating it. The simplest realistic case is that of a bare circular board excited by a driving port at a via-post at the centre of the circle. The circular symmetry means we can get an analytic solution for the fields and a closed form expression for the power-plane impedance; this allows us to get some physical insight into what is happening and helps build lumped-parameter models to assist in practical design problems without the nice symmetry. The analytic solution is also used to validate a numerical method for solving the field equations.

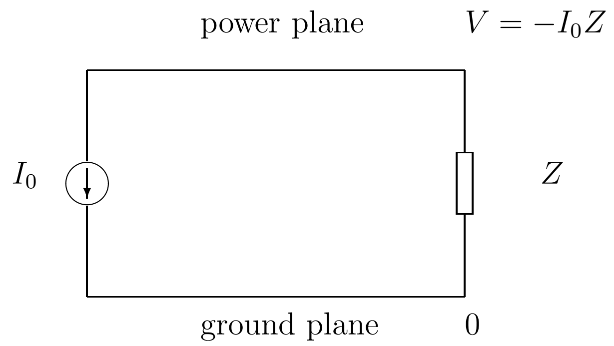

Figure 1: Power-plane noise

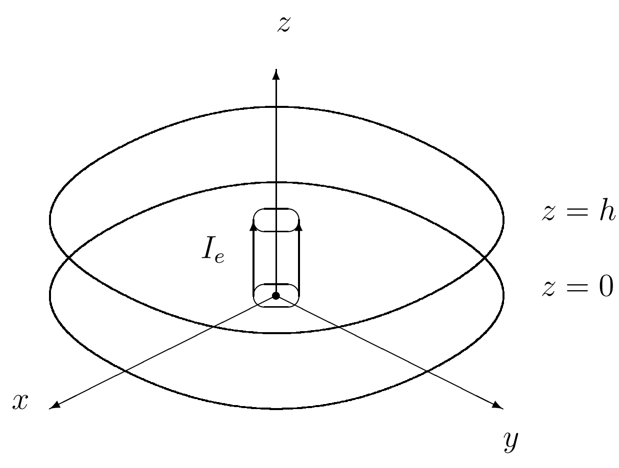

Figure 2: Geometry of a circular board

2 Impedance of a Bare Circular Board

2.1 The vector-Helmholtz equation in the gap

Consider the circular board shown in figure 2. The plane $z = 0$ is the ground plane. The power and ground planes are separated by a height h which is much smaller than the outer radius r_1. The electric field E in the gap between the planes is assumed to point purely along the z axis and to have no dependence on the height z. The source of the fields is an impressed axial excitation current $I_e = -I_0$ which flows up[1] the via post of small radius r_0 at the centre of the board. This produces a TM mode field; magnetic field H is azimuthal and the E-field is axial.

Maxwell's equations [1] for the electromagnetic field (E,H) are,

$$\nabla \times H = J + \frac{\partial D}{\partial t} \tag{1}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{2}$$

$$\nabla \bullet B = 0 \tag{3}$$

$$\nabla \bullet D = \rho \tag{4}$$

$$D = \epsilon E = \epsilon_0 \epsilon_r E \tag{5}$$

$$B = \mu H = \mu_0 \mu_r H \ . \tag{6}$$

The time-dependence of the fields is $\exp(i\omega t)$ so that, formally, $\partial/\partial t = i\omega t$. We set out to get a vector-Helmholtz equation for the H-field.

$$\begin{aligned}\nabla \times \nabla \times H &= \nabla(\nabla \bullet H) - \nabla^2 H \\ &= \nabla \times J + \nabla \times (i\omega\epsilon E) \\ &= \nabla \times J + i\omega\epsilon(-i\omega\mu H)\end{aligned}$$

There are no current (J) or charge (ρ) sources in the gap between the planes in the space outside the via post, so we get the vector-Helmholtz equation,

$$\nabla^2 H + k^2 H = 0 \tag{7}$$

where the wave-number $k = \omega\sqrt{\mu\epsilon}$.

[1] Although the mechanism for power plane noise is modeled by a current source switching a pulse of current going down a via-post, there is less chance of making a sign error when positive current flows along the positive z-axis.

2.2 Bessel's equation for the magnetic field

The small gap ($h \ll r_1$) between the planes means that the only non-zero fields are the E_z component of the electric field and the azimuthal component H_ϕ of the magnetic field. Furthermore, the circular-symmetry means that these components only depend on the radius as $E_z(r)$ and $H_\phi(r)$. The vector Helmholtz equation (7) then simplifies to,

$$\frac{d^2 H_\phi}{d(kr)^2} + \frac{1}{kr}\frac{dH_\phi}{d(kr)} + \left(1 - \frac{1}{(kr)^2}\right) H_\phi = 0 \tag{8}$$

which is Bessel's equation of order one [2]. In obtaining this equation, we have divided through by k^2 so that the radial dependence is in terms of the dimensionless variable kr.

We want solutions of equation (8) in terms of an outgoing wave propagating away from the via hole and an incoming wave reflected from the rim of the board; we pick the Hankel functions [2] $H_1^{(1)}$ and $H_1^{(2)}$ of the first and second kinds of order one. The general solution is,

$$H_\phi = \alpha H_1^{(1)}(kr) + \beta H_1^{(2)}(kr) \tag{9}$$

where α and β are constants which must be adjusted to fit the boundary conditions.

The asymptotic behaviour of the Hankel functions of order n as $kr \to \infty$ is [2],

$$\begin{aligned} H_n^{(1)}(kr) &\to \sqrt{\frac{2}{\pi kr}}\, e^{ikr - i\pi(2n+1)/4} \\ H_n^{(2)}(kr) &\to \sqrt{\frac{2}{\pi kr}} e^{-ikr + i\pi(2n+1)/4} \end{aligned} \tag{10}$$

so that, with the time-dependence of the fields of $e^{i\omega t}$, $H_1^{(1)}$ is the incoming wave reflected from the rim, and $H_1^{(2)}$ is the outgoing wave propagating away from the via post.

2.3 The boundary conditions on the fields

The circular symmetry ensures that the currents flow radially in the planes. Figure 3 shows how to relate the radial current density component K_r with units A/m to the magnetic field H_ϕ in the gap. Integrating the Maxwell equation (1) over the surface of the small loop in figure 3 and using Stokes'

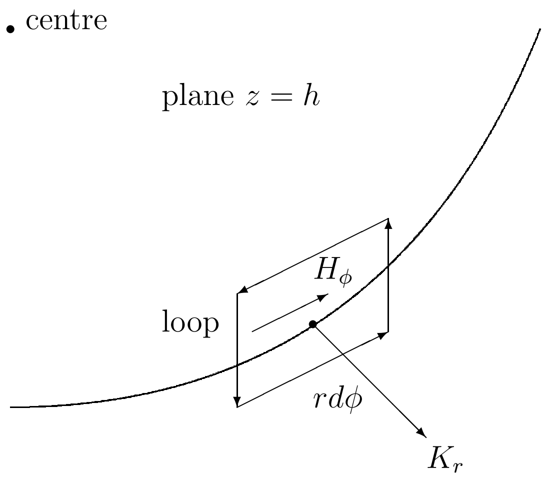

Figure 3: Current density on the top plane

theorem to transform to a line integral around the boundary of the loop gives,

$$\begin{aligned}\oint H \bullet dl &= \int J \bullet dS \\ H_\phi r d\phi &= +K_r r d\phi \ .\end{aligned}$$

The plus sign is because the calculation is done on the top plane so that the H-field only exists in the gap below the plane: only the bottom edge of the integration around the boundary of the loop sees a contribution from the H-field. Consequently,

$$H_\phi = K_r \tag{11}$$

and it's important to remember that K_r is the current density on the top plate; the bottom plate current density is $-K_r$.

Now we can get the coefficients α and β in equation (9). The current density must be zero at the rim of radius r_1. Hence,

$$0 = \alpha H_1^{(1)}(kr_1) + \beta H_1^{(2)}(kr_1) \ . \tag{12}$$

The current I_e flowing up the via post of radius r_0 supplies the current density $K_r(r_0)$ in the top plate (see figure 4). Therefore,

$$K_r(r_0) = \frac{I_e}{2\pi r_0} = H_\phi(r_0) \tag{13}$$

and so,

$$\frac{I_e}{2\pi r_0} = \alpha H_1^{(1)}(kr_0) + \beta H_1^{(2)}(kr_0) \ . \tag{14}$$

Solving equations (12) and (14) for α and β and plugging the results into (9) gives the H-field in the gap as,

$$H_\phi(r) = \frac{I_e}{2\pi r_0}\left(\frac{H_1^{(1)}(kr)H_1^{(2)}(kr_1) - H_1^{(2)}(kr)H_1^{(1)}(kr_1)}{H_1^{(1)}(kr_0)H_1^{(2)}(kr_1) - H_1^{(2)}(kr_0)H_1^{(1)}(kr_1)}\right) \ . \tag{15}$$

2.4 The electric field in the gap

The volume current density $J = 0$ in the gap outside the via post. Using the Maxwell equation (1), the electric field in the gap is,

$$i\omega\epsilon E = \nabla \times H \ = \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)e_z$$

where e_z is a unit vector along the z-axis. Dividing through by k,

$$i\sqrt{\frac{\epsilon}{\mu}}E_z = \ \frac{H_\phi}{kr} + \frac{dH_\phi}{dkr} \ . \tag{16}$$

In order to get a formula for the E-field in accordance with (16), we need to be able to differentiate the Hankel functions and simplify the results. This can be done with the aid of the recurrence relations [2],

$$2\frac{d}{dz}H_n^{(m)}(z) \ = \ H_{n-1}^{(m)}(z) - H_{n+1}^{(m)}(z) \tag{17}$$

$$\frac{2n}{z}H_n^{(m)}(z) \ = \ H_{n-1}^{(m)}(z) + H_{n+1}^{(m)}(z) \ . \tag{18}$$

Using (17) to evaluate the derivitive in (16) and then (18) to get rid of the Hankel functions of order $n = 2$, we obtain the electric field in the gap as,

$$i\sqrt{\frac{\epsilon}{\mu}}E_z(r) = \frac{I_e}{2\pi r_0}\left(\frac{H_0^{(1)}(kr)H_1^{(2)}(kr_1) - H_0^{(2)}(kr)H_1^{(1)}(kr_1)}{H_1^{(1)}(kr_0)H_1^{(2)}(kr_1) - H_1^{(2)}(kr_0)H_1^{(1)}(kr_1)}\right) \ . \tag{19}$$

2.5 The power-plane impedance

The potential difference between the planes at the position of the via post is $V = -E_z(r_0)h$ where h is the gap between the planes. So, evaluating (19) at r_0 and multiplying through by h gives,

$$-i\sqrt{\frac{\epsilon}{\mu}}V = \frac{I_e h}{2\pi r_0}\left(\frac{H_0^{(1)}(kr_0)H_1^{(2)}(kr_1) - H_0^{(2)}(kr_0)H_1^{(1)}(kr_1)}{H_1^{(1)}(kr_0)H_1^{(2)}(kr_1) - H_1^{(2)}(kr_0)H_1^{(1)}(kr_1)}\right) \ . \tag{20}$$

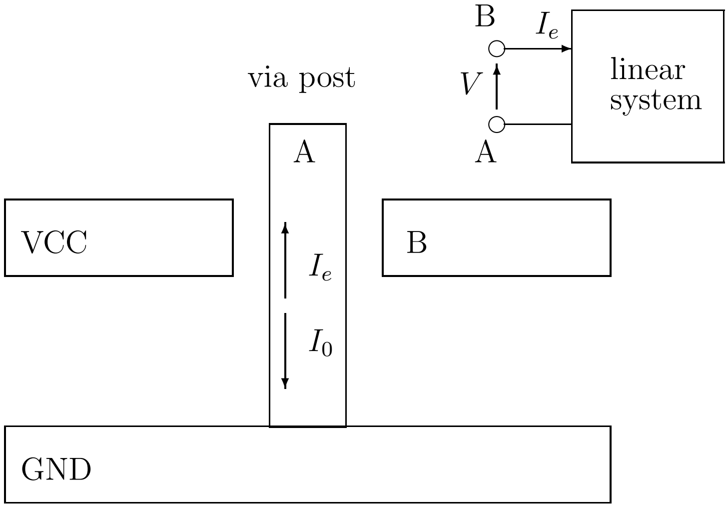

Figure 4: Via post port

Figure 4 shows how to think about the power-planes as a linear system which can be driven at a port formed by the pair of terminals A and B; A and B are physically identified as points on the via post and its surround. Conventionally, positive excitation current I_e goes into the top terminal B and out of the ground terminal A. Forming the ratio $Z = V/I_e$ in (20) gives the power-plane impedance of the system looking into the port formed by the via-post.

$$Z = \frac{V}{I_e} = \frac{1}{2\pi i}\sqrt{\frac{\mu}{\epsilon}}\frac{h}{r_0}\left(\frac{H_0^{(2)}(kr_0)H_1^{(1)}(kr_1) - H_0^{(1)}(kr_0)H_1^{(2)}(kr_1)}{H_1^{(1)}(kr_0)H_1^{(2)}(kr_1) - H_1^{(2)}(kr_0)H_1^{(1)}(kr_1)}\right) \tag{21}$$

2.6 Bare circular board example

2.6.1 How to run the program

Appendix A lists a Scilab [3] program called `vector-helmholtz.sci` which evaluates equation (21). In order to run this program, first edit the values set up on the program lines which are commented by a `?` symbol. The program can also calculate the effect of a large number of decoupling capacitors using the method described later in section 3.11.1, so in order to get the bare board impedance, we have to turn off the decoupling capacitor code with the switch,

```
bare_board_impedance=%T;
```

parameter	value	meaning
r0	125.0e-6 m	via-post radius
r1	100.0e-3 m	radius of board
h	150.0e-6 m	gap between planes
er	4.2	relative permittivity of gap
C	7.8nF	bare board capacitance
L	181pH	bare board inductance
f_{z1}	134MHz	first zero

Table 1: Parameters for a bare circular board

Table 1 shows the values that were set up for the example in this section[2]. Then, in the Scilab terminal type,

```
-->exec('vector-helmholtz.sci');
```

which runs the program. The program computes the impedance and stores it in the array `Z0` at a range of frequencies stored in the array `fvec`. A Bode plot of the magnitude $20 \log |Z|$ dB and phase $\angle Z$ deg versus the frequency is obtained by the following command.

```
-->bode(fvec,Z0);
```

The Bode plot in figure[3] 5 used the parameters in table 1. The initial frequency and the number of decades of frequency to compute and plot are set up by lines marked with a `?` symbol in the program.

2.6.2 The bare board at low frequencies

At low frequency in figure 5, the bare circular board looks like a parallel-plate capacitance of value,

$$C = \frac{\epsilon_0 \epsilon_r \pi r_1^2}{h} \tag{22}$$

and impedance,

$$Z = \frac{1}{sC} \ . \tag{23}$$

Using the values in table 1, we find $C = 7.8$nF and at 100kHz the impedance is $205\Omega\angle{-90}\,$deg which is 46dB in agreement with the computed value in figure 5.

[2] Only the values above the horizontal line in the table are input parameters, the values below the horizontal line are explained in section 2.6.3

[3] For now, ignore the trend line that goes through the pole-zero responses above 600MHz; it is the result of the circuit model described in section 2.6.3

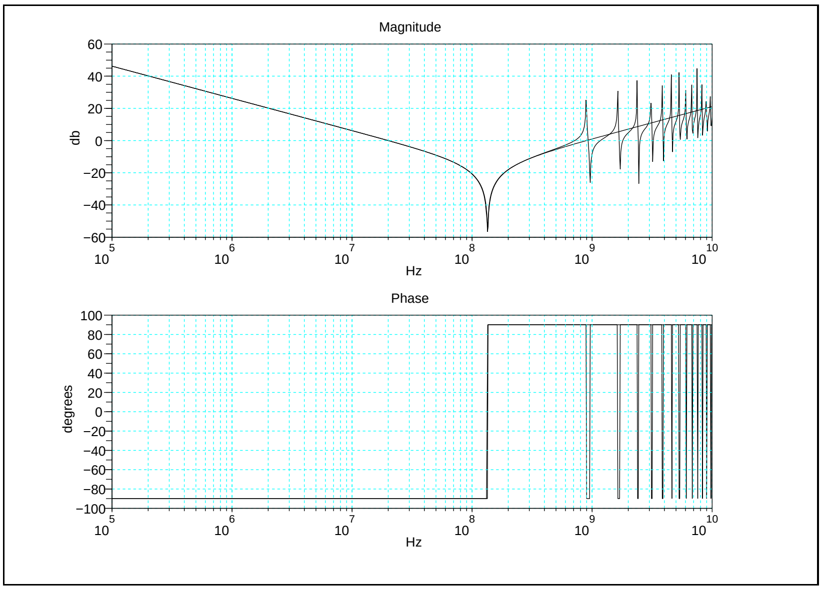

Figure 5: Power-plane impedance of bare circular board

2.6.3 The bare board at medium and high frequencies

Figure 5 shows that as the frequency increases, the bare board ceases to look like a capacitor and changes to look like an inductor. The change-over occurs at the position of the first zero of the impedance at 134MHz in figure 5. This effect of the first zero is captured by the circuit model in figure 6 which has impedance,

$$Z = sL + \frac{1}{sC} = \frac{1 + s^2 LC}{sC} \ . \tag{24}$$

The angular frequency of the first zero is at $1/\sqrt{LC}$. Plugging in the frequency of the first zero at 134MHz and the capacitance of the bare board $C = 7.8\text{nF}$ gives the equivalent inductance in the model as $L = 181\text{pH}$. In order to see how accurate this model is, as already mentioned, figure 5 overlays the field theory Bode plot with the Bode plot of the circuit model. We see that the circuit model agrees exactly with the field theory below 600MHz. Above 600MHz, the circuit model captures the trend of the bare board impedance. Of course, the circuit model is only accurate because we have set up the equivalent inductance L to reproduce the first zero of the field theory solution. Nevertheless, the circuit model is a useful device to help us understand the properties of the field solution.

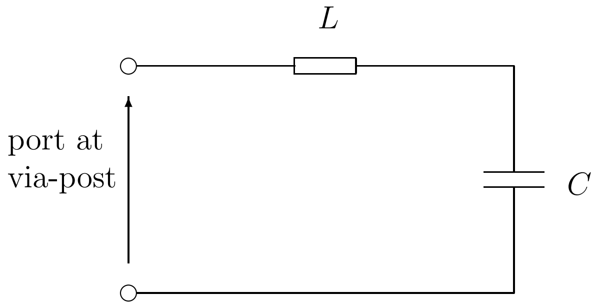

Figure 6: Circuit-model for a bare board

3 Numerical Method for Power Plane Impedance

3.1 The scalar Helmholtz equation for the E-field

Now we need to compute the power-plane impedance of a general circuit board which is not circularly symmetric and which has many via-posts which serve as ports to excite and decouple the board. The field is still a TM mode; the electric field still has only the E_z component, but lack of circular symmetry means that the H_x and H_y components of the magnetic field are non-zero. This means that it is easier to formulate the problem in terms of the E-field because it has only one component. From Maxwell's equations (2,1,4,5, 6) we obtain an inhomogeneous scalar Helmholtz equation for the E-field in the gap between the planes,

$$\nabla^2 E_z + k^2 E_z = i\omega\mu J_z \tag{25}$$

where J_z is the current density flowing (in the positive z-direction) in the via-posts.

3.2 Green's function for the scalar Helmholtz equation

The technique used to solve equation (25) is the Green's function method described in reference [2]. The basic idea is to find the field due to a filament current (the Green's function), and then to get the E-field due to the actual sources by making them out of filament currents and summing the Green's functions. The only problem is that the impressed currents in the via-posts are not the only sources; there will be charges induced at the edge of the board to stop the currents in the top and bottom plates flowing off the board. However, a clever use of Green's theorem in the plane will allow us to account for these induced sources using the same Green's function.

Firstly, we need a Green's function. The Green's function $G(x, y|x_0, y_0)$ is the E-field response at point (x, y) to a filament current at (x_0, y_0). It satisfies the wave equation (25) in the form,

$$\nabla^2 G + k^2 G = \delta(x - x_0)\delta(y - y_0) \tag{26}$$

where $\delta(x)$ is the Dirac delta function. The factor $i\omega\mu$ has been omitted from the source term; it will be put back when we come to add up the filament currents to produce the source at a via-post of finite radius. The solution of (26) is an outgoing wave from the source and so it is easiest to solve it in cylindrical coordinates (r, ϕ, z) with the filament current coincident with the z-axis. In these coordinates (26) becomes,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) + k^2 = \frac{\delta(r)}{2\pi r} \tag{27}$$

and outside the singularity at $r = 0$ it is Bessel's equation of order zero,

$$\frac{d^2 G}{d(kr)^2} + \frac{1}{kr}\frac{dG}{d(kr)} + G = 0 \ . \tag{28}$$

We want the solution to be an outgoing wave from the source. Our time dependence is $\exp(i\omega t)$ and so, in view of the asymptotic forms in set (10), our outgoing wave is proportional to the zeroth-order Hankel function of the second kind $H_0^{(2)}(kr)$. So, the Green's function is $\alpha H_0^{(2)}(kr)$ where α is a constant needed to match up with the strength of the source term in (27). In order to fix α, imagine a cylinder of height h and radius r with the filament on the axis of the cylinder. Integrating equation (27) over the volume of the cylinder, and using the divergence theorem to transform to an integral over the boundary of the cylinder,

$$\int (\nabla^2 G + k^2 G)dV = \oint \nabla G \bullet dS + k^2 \int G dV = \int \frac{\delta(r)}{2\pi r} dV \ .$$

As the radius of the cylinder $r \to 0$ the volume integral is zero and we are left with,

$$h\int_0^{2\pi} \frac{dG}{dr} r d\phi = h$$

and so we must have,

$$1 = 2\pi r \frac{dG}{dr} = 2\pi k r \alpha \frac{dH_0^2(kr)}{dkr} \text{ as } r \to 0 \ . \tag{29}$$

Now, the Hankel functions [2] are defined in terms of the Bessel (J_n) and Neumann (N_n) functions as,

$$\begin{aligned} H_n^{(1)}(z) &= J_n(z) + iN_n(z) \\ H_n^{(2)}(z) &= J_n(z) - iN_n(z) \ . \end{aligned} \tag{30}$$

The behaviour of the functions for a small argument is [2],

$$\lim_{z\to 0} J_n(z) = \frac{1}{n!}\left(\frac{z}{2}\right)^n \tag{31}$$

$$\lim_{z\to 0} N_n(z) = -\frac{(n-1)!}{\pi}\left(\frac{2}{z}\right)^n \text{ for } n > 0 \tag{32}$$

$$\lim_{z\to 0} N_0(z) = \frac{2}{\pi}(\ln z/2 + \gamma) \tag{33}$$

where $\gamma = 0.5772$ is the Euler-Mascheroni constant. The Neumann functions blow-up at the origin, so, upon plugging the form of N_0 into the condition (29), we find $\alpha = i/4$. So, after changing the source filament to a general position (x_0, y_0), the solution of equation (26) is the Green's function,

$$G(x, y|x_0, y_0) = \frac{i}{4} H_0^{(2)}\left(k\sqrt{(x-x_0)^2 + (y-y_0)^2}\right) \ . \tag{34}$$

3.3 Integral equation for the electric field

Green's theorem [2] relates a volume integral of two scalar functions f, g to a surface integral over the boundary of the volume.

$$\int (f\nabla^2 g - g\nabla^2 f)dV = \oint (f\nabla g - g\nabla f) \bullet dS \ . \tag{35}$$

We now take f as the electric field E_z which obeys the wave equation (25) and g as the Green's function G which is (34) and obeys (26). Using (25) and (26) to remove the terms $\nabla^2 E_z$ and $\nabla^2 G$ gives,

$$\int E_z \delta(x-x_0)\delta(y-y_0)dV - i\omega\mu \int G J_z dV = \oint (E_z \nabla G - G\nabla E_z) \bullet dS \ . \tag{36}$$

Let the integration be over the pancake-shaped volume V formed by the gap between the plates of the board. E_z and G do not depend on z, so that the gradients ∇E_z and ∇G have zero components in the z-direction. Consequently, the top and bottom surfaces of the pancake do not contribute to integral on the RHS of (36); the RHS integration is over the thin strip around the edge of the pancake. We can use figure 3 to work out the boundary condition to be applied to this edge strip. The figure was drawn for the circular board, but the edge and the loop shown are now regarded as being at some arbitrary point on the edge of a general board. The loop is tangential to the edge of the board; the boundary condition is that no current flows off the edge of the board, so the current component K_r in figure 3 - which is normal to the loop - must be zero. Equation (11) then shows that the tangential magnetic field component H_ϕ is also zero. It's then easy to use the Maxwell equation (2) to show that the electric field has zero gradient normal to the loop; ∇E_z has no components normal to the edge strip. Now we carry out the integrations in (36); the top and bottom surfaces of the pancake do not contribute and ∇E_z does not contribute to the edge of the pancake. Consequently,

$$h\int E_z\delta(x-x_0)\delta(y-y_0)dxdy - i\omega\mu h\int GJ_z dxdy = \; h\oint E_z\nabla G \bullet e_n|dl|$$

where we have put $dS = h|dl|e_n$ where $|dl|$ is the length of the line element vector at the edge of the board and e_n is the outward unit normal vector at the edge of the board. The volume integrals have also been changed to area integrals over a plane at constant z in the gap between the plates. Integrating the delta functions,

$$\begin{aligned} E_z(x_0,y_0) &= i\omega\mu\int G(x,y|x_0,y_0)J_z(x,y)dxdy \\ &\quad + \oint E_z(x,y)\nabla_{x,y}G(x,y|x_0,y_0)) \bullet e_n|dl| \end{aligned} \tag{37}$$

where $\nabla_{x,y}$ reminds us that the gradient operates on the source point with coords (x,y) and not on the field point with coords (x_0,y_0).

Now we need to substitute Green's function (34) into (37). In order to do this, we simplify the notation by using the position vector p for the point with coordinates (x,y). We need the gradient of Green's function,

$$\nabla_p G(p|p_0) = \frac{i}{4}\nabla_p H_0^{(2)}(k|p-p_0|) \; = \frac{i}{4}\frac{dH_0^{(2)}(k|p-p_0|)}{dk|p-p_0|}\frac{k(p-p_0)}{|p-p_0|} \tag{38}$$

The derivitive of the Hankel function can be found in terms of $H_1^{(2)}$ and $H_{-1}^{(2)}$ with the aid of recurrence relation (17). Then, by putting $n=0$ in recurrence relation (18), we get $H_{-1}^{(2)} = -H_1^{(2)}$: the gradient term simplifies to,

$$\nabla_p G(p|p_0) = -\frac{ik}{4}H_1^{(2)}(k|p-p_0|)\frac{(p-p_0)}{|p-p_0|} \; . \tag{39}$$

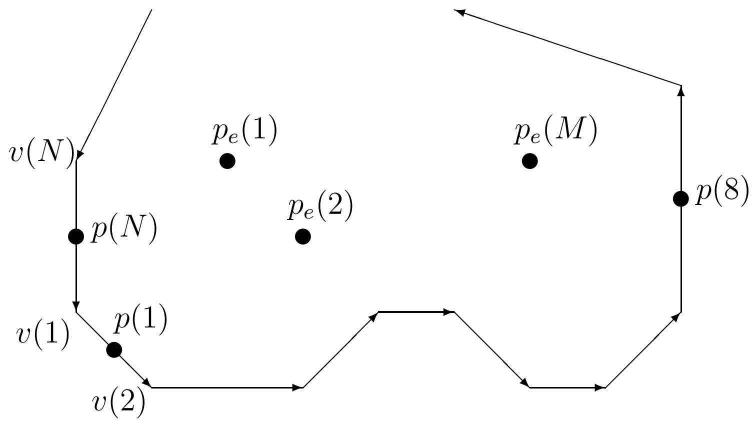

Figure 7: Boundary of the board and the via-posts

Now we can replace Green's function in (37) by the Hankel functions and we get,

$$\begin{aligned} E_z(p_0) &= -\frac{k}{4}\sqrt{\frac{\mu}{\epsilon}} \int H_0^{(2)}(k|p-p_0|)J_z(p)dA \\ &\quad -\frac{ik}{4} \oint E_z(p)H_1^{(2)}(k|p-p_0|)\ \frac{(p-p_0) \bullet e_n|dl|}{|p-p_0|} \end{aligned} \tag{40}$$

where the occurrence of ω has been replaced by the wave-number k from its definition following (7) and $dA = dxdy$. Equation (40) is an integral equation for the unknown electric field $E_z(p_0)$ in the gap when excited by the impressed current density J_z in the via-posts and the induced charges on the edge of the board. The boundary integral (second term on the RHS) represents the effect of the induced charges; they stop the plate current flowing off the edge of the board.

Appendix B lists a Scilab [3] program called `method-of-moments-v5.sci` which computes the power-plane impedance of a general board - with decoupling capacitors - by numerically solving integral equation (40). The following sections describe the numerical method and note where the relevant computations take place in the program by reference to the listing in the appendix.

3.4 Solution for the electric field on the boundary

The first task is to use the integral equation (40) to solve for the electric field on the boundary. The boundary of the board is a polygon with vertices $v(1), \ldots, v(N)$ as shown in figure 7. In the program (appendix B), the boundary vertices are specified by the $N \times 2$ matrix `v` so that the rth vertex is the row vector `v(r,:)`. The electric field is assumed to be constant on each boundary segment; we use the midpoints of each boundary segment in order to evaluate the boundary electric field. These midpoints are shown as $p(r)$ in figure 7 and in the program they are represented by the $N \times 2$ matrix `p`.

The excitation is provided by the impressed currents in the via-posts. There are M via-posts at the points $p_e(1), \ldots, p_e(M)$ (see figure 7). In the program, the coordinates of the via-posts are in the $M \times 2$ matrix `pe`. We approximate the current density $J_z(p)$ by delta function filament currents at the via-posts,

$$J_z(p) = \sum_{r=1}^{M} I_e(r)\delta(p - p_e(r)) \tag{41}$$

where $I_e(r)$ is the actual current in Amp into the rth port. Note that $I_e(r)$ is the current travelling up the via-post and into the port as shown in figure 4.

We get a matrix equation for the boundary electric fields by taking the field point p_0 in the integral equation (40) on the segments and approximating the continuous boundary integral by a discrete sum over the segments.

$$\begin{aligned} E_z(p(r)) &= -\frac{k}{4}\sqrt{\frac{\mu}{\epsilon}} \sum_{s=1}^{M} H_0^{(2)}(k|p_e(s) - p(r)|)I_e(s) \\ &\quad -\frac{ik}{4}\sum_{s=1}^{N} E_z(p(s))H_1^{(2)}(k|p(s) - p(r)|)\ \frac{(p(s) - p(r)) \bullet e_n(s)|dl(s)|}{|p(s) - p(r)|} \end{aligned}$$

This looks more like a matrix equation if we reverse the order of the point differences,

$$\begin{aligned} E_z(p(r)) &= -\frac{k}{4}\sqrt{\frac{\mu}{\epsilon}} \sum_{s=1}^{M} H_0^{(2)}(k|p(r) - p_e(s)|)I_e(s) \\ &\quad +\frac{ik}{4}\sum_{s=1}^{N} H_1^{(2)}(k|p(r) - p(s)|)\ \frac{(p(r) - p(s)) \bullet e_n(s)|dl(s)|}{|p(r) - p(s)|}E_z(p(s)) \ . \end{aligned} \tag{42}$$

Now we can define a $N\times M$ matrix $C(N\times M)$ and a $N\times N$ matrix $D(N\times N)$

with components[4],

$$C_{rs}(N \times M) \;=\; -\frac{k}{4}\sqrt{\frac{\mu}{\epsilon}}\; H_0^{(2)}(k|p(r) - p_e(s)|) \tag{43}$$

$$D_{rs}(N \times N) \;=\; \frac{ik}{4} H_1^{(2)}(k|p(r) - p(s)|)\; \frac{(p(r) - p(s)) \bullet e_n(s)|dl(s)|}{|p(r) - p(s)|} \tag{44}$$

so that (42) can be written as,

$$E_z(p(r)) = \sum_{s=1}^{M} C_{rs}(N \times M) I_e(s) \; + \sum_{s=1}^{N} D_{rs}(N \times N) E_z(p(s)) \; . \tag{45}$$

In the program (appendix B), the $C(N \times M)$ matrix (43) is written `CNxM` and the $D(N \times N)$ matrix (44) is `DNxN`. Physically, $C(N \times M)$ is the transfer matrix for boundary fields due to port currents whilst $D(N \times N)$ is the transfer matrix for boundary fields due to boundary fields. If we write the boundary fields as a $N \times 1$ matrix $E_z(N \times 1)$ (program: `EzNx1`), and the port currents as a $M \times 1$ matrix I_e (program: `IeMx1`) then (45) is,

$$E_z(N \times 1) = C(N \times M) I_e + D(N \times N) E_z(N \times 1)$$

which has the solution,

$$E_z(N \times 1) = (1 - D(N \times N))^{-1} C(N \times M) I_e \; . \tag{46}$$

In principle, this solves the field problem, because - once the impressed port currents have been given and the boundary fields determined - we can evaluate the E-field anywhere in the gap by a straightforward use of equation (40). However, in practice, the crude approximation of the integrals in the $D(N \times N)$ matrix (44) is not good enough. Firstly, the diagonal elements $D_{rr}(N \times N)$ in formula (44) blow up so they cannot be used. Secondly, it turns out that when the wave-number k is low, the off-diagonal elements from formula (44) are not accurate enough and the calculated power-plane impedance is not physically sensible at low k. The next section fixes the problem of the diagonal elements of the $D(N \times N)$ matrix by taking more care with the approximate integration over the boundary segments. Then, section 3.6 fixes the problem at low k by brute-force numerical integration.

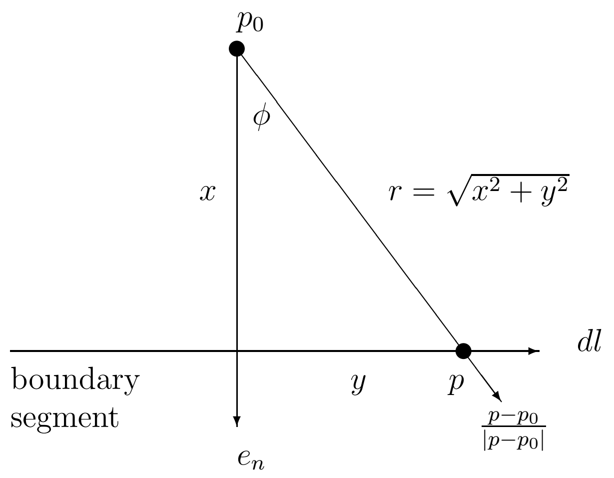

Figure 8: Geometry for diagonal elements of the D-matrix

3.5 Diagonal elements of the D-matrix

From equation (40), the diagonal elements of the $D(N \times N)$ matrix essentially require the computation of the integral,

$$I = \int H_1^{(2)}(k|p - p_0|) \frac{(p - p_0) \bullet e_n |dl|}{|p - p_0|} \tag{47}$$

where the source point p is integrated over a boundary segment and the field point p_0 is evaluated at the mid-point of the same boundary segment. In order to avoid the singularity of the integrand, we place the field point a small distance away from the boundary, and then take the limiting case as the field point p_0 is brought up to the midpoint of the boundary segment along the normal vector e_n. The geometry for this limiting process is shown in figure 8. Most of the contribution to the integral occurs near the singularity where $p - p_0$ is small. Hence, we can use the limiting form of the Hankel function for a small argument from equations (30,31,32).

$$\lim_{z \to 0} H_1^{(2)}(z) = \frac{z}{2} + \frac{2i}{\pi z} \tag{48}$$

The integral (47) is now,

$$I \approx \int dy \left(\frac{kr}{2} + \frac{2i}{\pi kr} \right) \cos\phi \; = \frac{kx|dl|}{2} + \frac{4i}{\pi k} \tan^{-1} \left(\frac{|dl|}{2x} \right) . \tag{49}$$

[4]It is helpful to label the matrices by their shape, because the C and D matrices both turn up in two shapes - one for computing the E-field on the boundary and the other for computing the E-field at the ports.

Now let the field point p_0 move onto the midpoint of the boundary segment by setting $x \longrightarrow 0$ in figure 8; the result is $I = 2i/k$. Upon restoring the constant factor $ik/4$ that appears in the matrix element (44) - remembering the sign change in the direction cosine factor that occurred in the passage from (40) to (44) - our new treatment of the diagonal elements of the $D(N \times N)$ matrix yields,

$$D_{rr}(N \times N) = \frac{1}{2} \ . \tag{50}$$

3.6 The D-matrix at low k

It has already been mentioned that the use of the approximation (44) for the off-diagonal elements of the $D(N \times N)$ matrix gave unphysical values for the power-plane impedance at low wave-number k; the board should look like a capacitor at low frequency, but in fact it looked like an inductor. This problem was solved by using Scilab's canned numerical integration routine `intg` to evaluate the off-diagonal elements[5] as,

$$D_{rs}(N \times N) = \frac{ik}{4} \int H_1^{(2)}(k|p(r) - p(s)|) \frac{(p(r) - p(s)) \ \bullet e_n |dl|}{|p(r) - p(s)|} \ . \tag{51}$$

However, it is quite time-consuming to numerically integrate (51) for all the diagonal elements at each wavenumber in order to get the frequency response of the power-plane impedance. Consequently, a cut-off frequency `fcutoff` was included in the program (appendix B); below the cut-off, numerical integration of (51) was used for the diagonal elements of the $D(N \times N)$ matrix, whilst above the cut-off, approximation (44) was used for the diagonal elements. The diagonal elements always use (50) regardless of the frequency.

At this stage, we can calculate the $C(N \times M)$ and $D(N \times N)$ matrices at all wave-numbers k and then plug them into (46) to get the electric field $E_z(N \times 1)$ on the boundary in terms of the impressed port currents I_e. The next section explains how to calculate the electric field in the gap between the planes.

3.7 Solution for the electric field in the gap

Once the E-field on the boundary is known, we can calculate the E-field anywhere in the gap between the planes using (40). The interesting case is to calculate the E-field at the ports formed by the via-posts; the port fields will be needed later in order to solve for the power-plane impedance.

[5]The diagonal elements are still given by equation (50) because the canned integration routine cannot handle the singularity in the integrand of the diagonal elements.

The matrix equations for the port fields are similar to equation (42) except that the field point p_0 is taken on the rth port position $p_e(r)$ instead of the boundary position $p(r)$.

$$\begin{aligned} E_z(p_e(r)) &= -\frac{k}{4}\sqrt{\frac{\mu}{\epsilon}}\ \sum_{s=1}^{M} H_0^{(2)}(k|p_e(r)-p_e(s)|)I_e(s) \\ &\quad +\frac{ik}{4}\sum_{s=1}^{N} H_1^{(2)}(k|p_e(r)-p(s)|)\ \frac{(p_e(r)-p(s))\bullet e_n(s)|dl(s)|}{|p_e(r)-p(s)|}E_z(p(s))\ . \end{aligned} \tag{52}$$

This is a matrix equation,

$$E_z(p_e(r)) = \sum_{s=1}^{M} C_{rs}(M\times M)I_e(s)\ +\sum_{s=1}^{N} D_{rs}(M\times N)E_z(p(s)) \tag{53}$$

analogous to equation (45). The $M\times M$ matrix $C(M\times M)$ and the $M\times N$ matrix $D(M\times N)$ defined by,

$$C_{rs}(M\times M) = -\frac{k}{4}\sqrt{\frac{\mu}{\epsilon}}\ H_0^{(2)}(k|p_e(r)-p_e(s)|) \tag{54}$$

$$D_{rs}(M\times N) = \frac{ik}{4}H_1^{(2)}(k|p_e(r)-p(s)|)\ \frac{(p_e(r)-p(s))\bullet e_n(s)|dl(s)|}{|p_e(r)-p(s)|} \tag{55}$$

are analogous to the $C(N\times M)$ and $D(N\times N)$ matrices for the boundary fields in (43,44) except that they are different shapes. In the program (appendix B), the $C(M\times M)$ transfer matrix (54) for port fields due to port currents is `CMxM` and the $D(M\times N)$ transfer matrix (55) for port fields due to boundary fields is `DMxN`. If we write the port fields as a $M\times 1$ matrix $E_z(M\times 1)$ then (53) is,

$$E_z(M\times 1) = C(M\times M)I_e + D(M\times N)E_z(N\times 1)\ . \tag{56}$$

However, there is a problem; the diagonal elements $C_{rr}(M\times M)$ in (54) blow-up because the approximation is too crude to account correctly for the contribution of a port current to its own port field. The next section fixes this problem with a careful treatment of the diagonal elements.

3.8 Diagonal elements of the C matrix

In order to get a better approximation to the diagonal elements of the $C(M\times M)$ matrix, we return to the basic integral equation (40) and consider the first integral on the RHS which evaluates the E-field due the port current density $J_z(p)$. Take the field point p_0 at the centre of a via-post (port) of radius r_0. Assume that the current I_e in the via-post is uniformly spread over the cross-section of the via-post. The current density in the via-post is,

$$J_z = \frac{I_e}{\pi r_0^2} \ . \tag{57}$$

The field response to a unit impressed current $I_e = 1$ is essentially,

$$\int H_0^{(2)}(k|p - p_0|) J_z(p) dA = \int_0^{r_0} H_0^{(2)}(kr) \frac{2\pi r dr}{\pi r_0^2}$$
$$= \frac{2}{(kr_0)^2} \int_0^{kr_0} H_0^{(2)}(u) u du \approx 1 - \frac{2i}{\pi}(\ln(kr_0/2) + \gamma - 0.5) \tag{58}$$

where the limiting form (30,31,33) of the Hankel function for $kr_0 << 1$ has been used in the integration. The diagonal elements of the $C(M \times M)$ matrix are now,

$$C_{rr}(M \times M) = -\frac{k}{4}\sqrt{\frac{\mu}{\epsilon}} \left(1 - \frac{2i}{\pi}(\ln(kr_0/2) + \gamma - 0.5)\right) \ . \tag{59}$$

Summarising, the E-field at the ports is calculated using matrix equation (56), the off-diagonal elements of the $C(M \times M)$ matrix use (54) whilst the diagonal elements use (59), the $D(M \times N)$ matrix uses (55).

Note that in the program (appendix B), the C and D matrices of both shapes are all computed in the function `port_matrices`.

3.9 Circuit board as a linear M-port

Substitute the solution (46) for the boundary fields into equation (56) for the port fields. We get the solution for the port fields in terms of the impressed port currents.

$$E_z(M \times 1) = (C(M \times M) + \ D(M \times N)(1 - D(N \times N))^{-1} C(N \times M)) I_e$$

The voltage across a via-post is $-hE_z$. Upon multiplying through by $-h$ we get,

$$V = -h(C(M \times M) + D(M \times N)(1 - D(N \times N))^{-1} C(N \times M)) I_e \tag{60}$$

where V is the $M \times 1$ column vector of port voltages. The board now looks like the linear M-port system shown in figure 9 characterized by the $M \times M$ impedance matrix,

$$Z_c = -h(C(M \times M) + D(M \times N)(1 - D(N \times N))^{-1} C(N \times M)) \ . \tag{61}$$

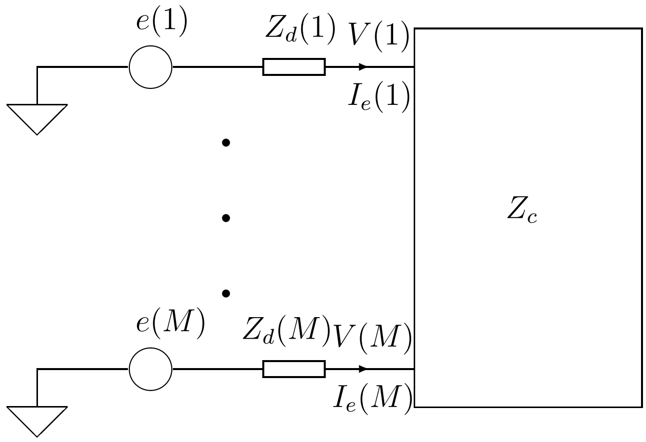

Figure 9: Circuit board as a M-port system

Although section 1 attributed the mechanism for exciting the board to pulsed current sources formed by the totem-pole transistors of chips, we are free to excite the board by an equivalent voltage source. This is the way the excitation is represented in figure 9; each port is excited by a voltage source formed by an EMF generator in series with a discrete impedance. The EMF generators are arrayed into a $M \times 1$ column vector e and the discrete impedances form a $M \times M$ diagonal impedance matrix Z_d. The matrix equation for the discrete components exciting the board is then,

$$V = e - Z_d I_e \tag{62}$$

and the response of the board is,

$$V = Z_c I_e \tag{63}$$

which is the same as equation (60). Combining (62) and (63) gives the following solution for the port currents due to the interaction of the board with the discrete components connected to its ports.

$$I_e = (Z_c + Z_d)^{-1} e \ . \tag{64}$$

The impedance of the bare board looking into port 1 (say) is found by putting the impressed port currents as the column vector,

$$I_e = [1, 0, \ldots, 0]^T$$

where the superscript T denotes the matrix transpose. The port voltages are given by (63) and so the voltage at port 1 is just the $Z_c(1,1)$ matrix element. Dividing by the unit current into port 1 shows that the bare board impedance looking into port 1 is given by the $Z_c(1,1)$ matrix element.

The impedance of the combined system formed by the board plus its decoupling capacitors looking into port 1 (say) is found by setting up the impedance matrix of the discretes with $Z_d(1) = 0$ and all the other elements on the diagonal filled by the impedances of the decoupling capacitors connected to ports $2, \ldots, M$. All the EMF generators are set to zero except for $e(1) = 1$. The port currents are then given by equation (64). The current into port 1 is then known from $I_e(1)$ and the impedance looking into port 1 is the ratio $e(1)/I_e(1)$.

3.10 Validation of the numerical method

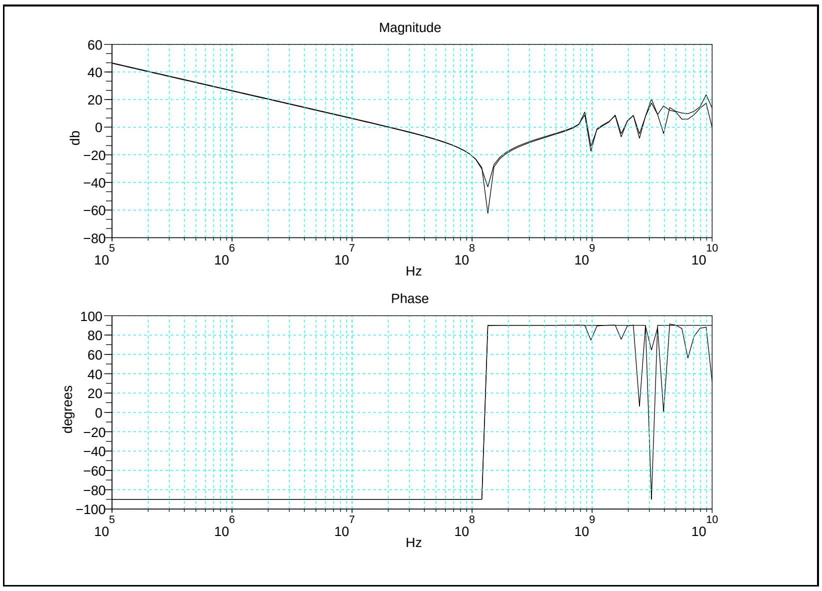

Figure 10: Analytic and numerical solns for a bare circular board

In order to check that the numerical method is sensible we compare it against the analytic solution for a circular board in section 2. The numerical method `method-of-moments-v5.sci` in appendix B is set up in a similar way to the program `vector-helmholtz.sci` described in section 2.6.1. The parameters of the board are set up in agreement with those in table 1. In addition, the switches,

```
board_shape=circular_board;
N=16;//No of boundary edges
calc_impedance=%T;
plot_bare_board_impedance=%T;
plot_power_plane_impedance=%F;
plot_boundary_and_ports=%F;
```

are set up to automatically generate 16 boundary vertices for a circular board and to only plot the bare board impedance. The switches to plot the power-plane impedance and to plot the boundary and ports are turned off (false); the program is set up with lots of ports with decoupling capacitors and we are not interested in the impedance of the decoupled board at this stage. The Bode plot - generated automatically upon running the program - is shown in figure 10 overlaid with the Bode plot of the analytic solution for the bare circular board which was shown earlier[6] in section 2.6. The numerical solution only deviates from the analytical solution in the depth of the first zero at 134MHz and above 3.5GHz; the numerical method seems to work when calculating the impedance of a bare board.

3.11 Another validation of the numerical method

We are also able to check that the numerical method works for the case of a board with decoupling because it is possible to get an analytic solution for a circular board populated with a large number of decoupling capacitors.

3.11.1 Analytic solution for a board with decoupling capacitors

Consider a large number M of decoupling capacitors placed uniformly on the circular board studied in section 2. For simplicity, we suppose that they are all of the same species. The admittance of a single discrete decoupling capacitor is,

$$Y_d = \frac{1}{Z_d} = \frac{1}{1/sC_d + sL_{esl} + R_{esr}} = \frac{sC_d}{s^2 L_{esl} C_d + sC_d R_{esr} + 1} \tag{65}$$

where C_d is the nominal capacitance, L_{esl} is the effective series inductance and R_{esr} is the effective series resistance. Let's use circuit theory to study the radial voltage and current. The diagram in figure 11 represents a thin

[6] The analytic solutions in figures 10 and 5 are the same, but figure 5 was plotted with many more data points in order to capture the pole-zero responses. Fewer data points were used in figure 10 because the numerical solution takes a few minutes to compute.

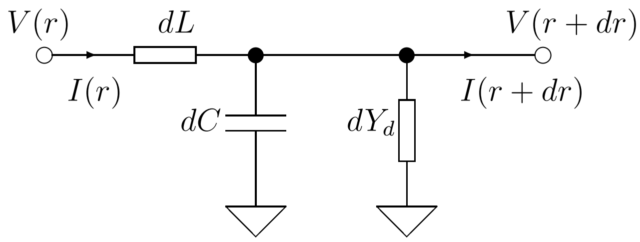

Figure 11: Circuit relations on a thin annulus of the board

ring at radius r of width dr. The board area of this ring is $2\pi r dr$ and so the admittance of the discrete decoupling capacitors inside the ring is,

$$dY_d = M \frac{2\pi r dr}{\pi r_1^2} Y_d \ . \tag{66}$$

The capacitance of the ring itself is,

$$dC = \frac{\epsilon 2\pi r dr}{h} \ . \tag{67}$$

The self-inductance dL of the ring can be obtained from equation (11) or (13). The magnetic flux linking the ring is,

$$d\Phi = \mu H_\phi h dr = \frac{\mu I}{2\pi r} h dr \tag{68}$$

and so the self-inductance is,

$$dL = \frac{\mu h dr}{2\pi r} \ . \tag{69}$$

The circuit relations are,

$$\begin{aligned} 0 &= V(r) - V(r+dr) - (i\omega dL) I(r) \\ I(r) &= (i\omega dC) V(r) + dY_d V(r) + I(r+dr) \end{aligned}$$

We get differential equations,

$$\begin{aligned}\frac{dV}{dr} &= -i\omega\frac{dL}{dr}I = -i\omega\frac{\mu h}{2\pi r}I \\ \frac{dI}{dr} &= -\left(i\omega\frac{dC}{dr} + \frac{dY_d}{dr}\right)V \;= -i\omega\frac{2\pi r}{h}\left(\epsilon + \frac{MhY_d}{i\omega\pi r_1^2}\right)V\ .\end{aligned}$$

Define the complex frequency-dependent effective permittivity,

$$\epsilon_{eff}(\omega) = \epsilon + \frac{MhY_d}{i\omega\pi r_1^2} \;= \epsilon + \frac{MC_d h/(\pi r_1^2)}{1 + i\omega C_d R_{esr} - \omega^2 L_{esl} C_d}\ . \tag{70}$$

The differential equations are now,

$$\frac{dV}{dr} = -i\omega\mu\frac{h}{2\pi r}I \tag{71}$$

$$\frac{h}{2\pi r}\frac{dI}{dr} = = -i\omega\epsilon_{eff}V\ . \tag{72}$$

By eliminating V from the above pair, we get a differential equation for $I/(2\pi r)$. From (68), we note that the azimuthal magnetic field is $H_\phi = I/(2\pi r)$. This equation for the magnetic field is,

$$\frac{d^2H_\phi}{dr^2} + \frac{1}{r}\frac{dH_\phi}{dr} \;+ \left(k_{eff}^2 - \frac{1}{r^2}\right)H_\phi = 0 \tag{73}$$

where the complex effective wave-number is,

$$k_{eff}^2 = \omega^2\mu\epsilon_{eff}\ . \tag{74}$$

Equation (73) is Bessel's equation of order one that also governed the magnetic field for the bare circular board in equation (8); the only difference is that (73) has replaced the ordinary wave-number k by the complex effective wave-number k_{eff}. So, in order to obtain the impedance of a circular board with a large number of decoupling capacitors, we just have to take the bare board solution (21) and make the replacement $k \to k_{eff}$ and $\epsilon \to \epsilon_{eff}$ to obtain,

$$Z = \frac{1}{2\pi i}\sqrt{\frac{\mu}{\epsilon_{eff}}}\frac{h}{r_0}\;\left(\frac{H_0^{(2)}(k_{eff}r_0)H_1^{(1)}(k_{eff}r_1)\; - H_0^{(1)}(k_{eff}r_0)H_1^{(2)}(k_{eff}r_1)}{H_1^{(1)}\;(k_{eff}r_0)H_1^{(2)}(k_{eff}r_1) - H_1^{(2)}(k_{eff}r_0)\;H_1^{(1)}(k_{eff}r_1)}\right)\ . \tag{75}$$

The Scilab program `vector-helmholtz.sci` (appendix A) evaluates the decoupling solution (75) when the bare board solution (21) is turned off with the program switch `bare_board_impedance=%F`.

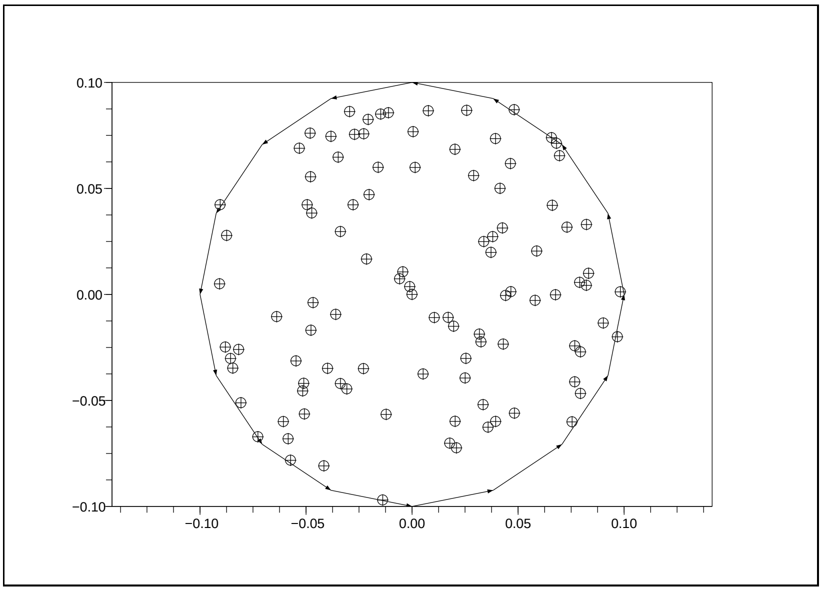

Figure 12: Decoupling capacitors for the numerical solution

parameter	value	meaning
r0	125.0e-6 m	via-post radius
r1	100.0e-3 m	radius of board
h	150.0e-6 m	gap between planes
er	4.2	relative permittivity of gap
M	100	no. of decoupling capacitors
Cd	100nF	capacitance per device
Lesl	1.5nH	effective series inductance per device
Resr	60mΩ	effective series resistance per device
C	7.8nF	bare board capacitance
L	181pH	bare board inductance
L_e	15pH	effective decoupling inductance
C_e	10μF	effective decoupling capacitance
$1/(2\pi\sqrt{L_e C})$	465MHz	pole

Table 2: Parameters for a circular board with decoupling

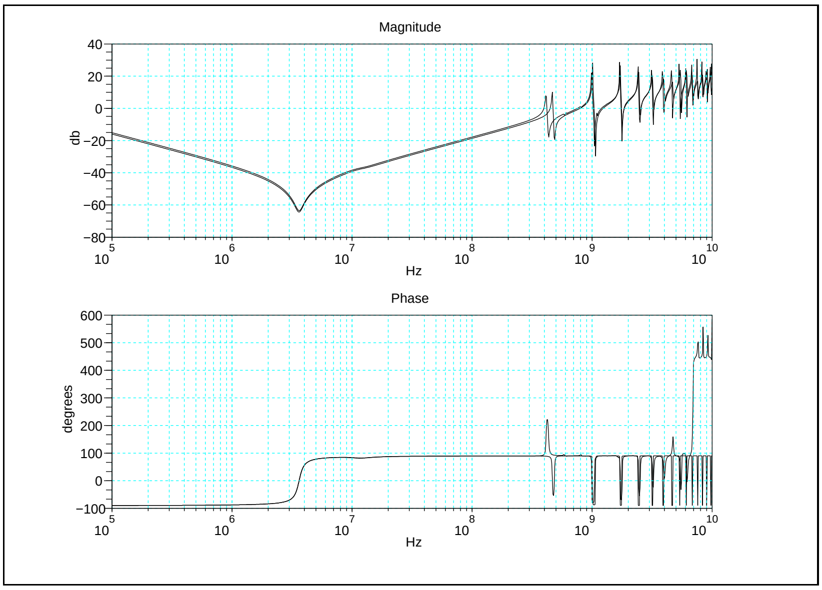

Figure 13: Analytic and numerical solns with decoupling caps

3.11.2 Analytic and numerical solutions for decoupling

Figure 13 overlays the Bode plots for the analytic and numerical solutions for a circular board with 100 100nF decoupling capacitors. The parameters for the board are in table 2. Figure 12 shows the random port positions used for the run of the numerical program. In the numerical program, the port positions are picked randomly with the switch `board_shape=circular_board`; the analytic solution uses the distributed approximation for the decoupling capacitors. The solutions agree except for a small discrepancy[7] in the position of the pole-zero pair at 400MHz. Given that the analytic solution replaces the discrete capacitors by a distributed approximation, the agreement seems quite good and gives additional confidence in the fidelity of the numerical solution.

4 Circuit Model for Power Plane Impedance

In this section we will use insights gained from the numerical field solution to explain the shape of the power plane impedance Bode plot in terms of circuit elements. This understanding is helpful when designing an electronic

[7] The phase discrepancy beyond 7GHz seems to be because Scilab's Bode plot has added 360 deg to the phase.

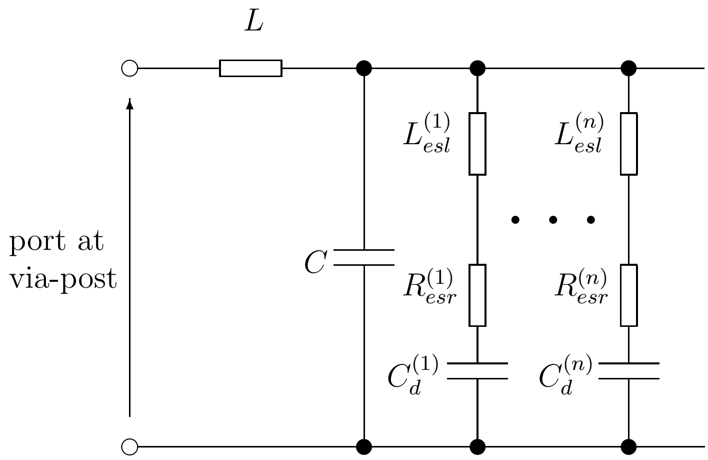

Figure 14: Circuit model for a decoupled board

system.

4.1 Circuit model for a decoupled board

Consider a board with several species of decoupling capacitors. We'll label the capacitor species by a superscript in brackets to remind us that it is not a power. The kth species has $M^{(k)}$ capacitors of nominal value $C_d^{(k)}$ connected in parallel and there are $k = 1, \ldots, n$ different species. Given the success of the circuit model of figure 6 which - see figure 5 - correctly produces the trend line of the power plane impedance for a bare circular board, then attaching the decoupling capacitors in the way shown in figure 14 seems a reasonable guess for a circuit model for a board with decoupling capacitors. The impedance Z_d of a single discrete decoupling capacitor was given in equation (65), so the effective impedance of $M^{(k)}$ capacitors connected in parallel to form the impedance for the kth species is,

$$Z_e^{(k)} = \frac{Z_d^{(k)}}{M^{(k)}} = \frac{1}{sM^{(k)}C_d^{(k)}} + \frac{sL_{esl}^{(k)}}{M^{(k)}} + \frac{R_{esr}^{(k)}}{M^{(k)}} = \frac{1}{sC_e^{(k)}} + sL_e^{(k)} + R_e^{(k)} \ . \quad (76)$$

The impedance of the network of figure 14, with all the species in parallel is,

$$Z = sL + \frac{1}{sC + \frac{1}{Z_e^{(1)}} + \ldots + \frac{1}{Z_e^{(n)}}} \ . \quad (77)$$

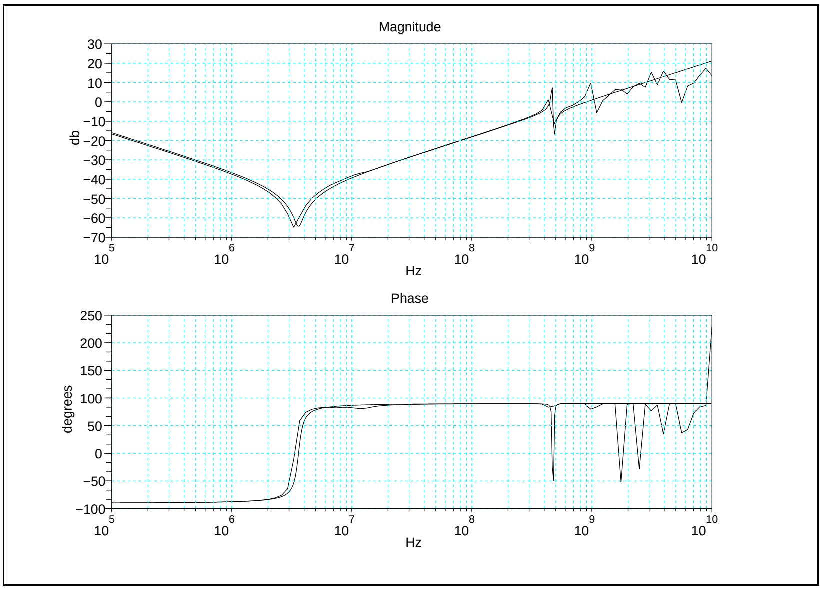

Figure 15: Circuit model vs numerical soln for decoupled circular board

For simplicity, let's use just one species of decoupling capacitors in order to compare the circuit model of equation (77) with the numerical field solution (program in appendix B). Table 2 lists the parameters for a circular board decoupled with a single species of 100 100nF capacitors; there is only a single species so we do not need the species superscript. The Bode plot of the power plane impedance is shown in figure 15. The circuit model impedance has been overlaid with the numerical solution for comparison. The agreement is quite good. As in the case of the bare board, the circuit model only agrees with the field solution because we have used the equivalent inductance $L = 181$pH that comes from the field solution for the bare board in section 2.6.3.

4.2 Qualitative features of the power plane impedance

Given that the circuit model appears to be a reasonable description of the system, let's study the qualitative shape of the power plane impedance when a board is decoupled with several species of capacitors.

At the lowest frequencies, the board will just look like a capacitor formed by all of the decoupling capacitances and the bare board capacitance in parallel. As the frequency increases, the individual species will resonate.

The kth species resonates at,

$$s^2 = -\frac{1}{L_e^{(k)} C_e^{(k)}} \ . \tag{78}$$

The shape of the impedance curve in this region depends on the species in a way which is too complicated to analyse in words. However, ignoring $R_{esr}^{(k)}$, when the kth species hits resonance, equation (76) shows that there is a zero in the species impedance $Z_e^{(k)}$. Therefore the denominator of the decoupling term in (77) blows up because $Z_e^{(k)} = 0$ and so the overall impedance is $Z = sL$. In other words, when a species resonates, it short circuits all the parallel connected elements in figure 14 and we are left with the bare board inductance. So, regardless of the detailed shape of the impedance, every time a species hits resonance, the overall impedance samples the bare board inductive line sL. If there are many species of capacitors, the overall impedance will sample the bare board inductive line sL at many points: the impedance will track the inductive line with some excursions on either side.

Now let's consider what happens when the frequency is above the self-resonances of the species capacitors. If the smallest capacitor species is 1nF with a parasitic inductance per device of 1.5nH, the species resonates at 130MHz. So, above 130MHz, the decoupling capacitors all look like inductors. The board impedance from equation (77) is now,

$$Z = sL + \frac{1}{sC + \frac{1}{s}\left(\frac{1}{L_e^{(1)}} + \ldots + \frac{1}{L_e^{(n)}}\right)} \ . \tag{79}$$

This equation shows that the individual species are no longer distinct and their effect can be handled by just considering the effective inductance,

$$\frac{1}{L_e} = \frac{1}{L_e^{(1)}} + \ldots + \frac{1}{L_e^{(n)}} \tag{80}$$

of all the decoupling capacitors of all species in parallel. In terms of this effective inductance, equation (79) is,

$$Z = sL + \frac{1}{sC + \frac{1}{sL_e}} = \frac{sL + sL_e + s^3 L L_e C}{1 + s^2 L_e C} \ . \tag{81}$$

Immediately above the highest decoupling self-resonance, $|s^2 L_e C| < 1$ because the bare board capacitance C is usually small (7.8nF in table 2) and L_e is always less than the inductance $L_e^{(n)}$ for the highest resonant species. In this regime, equation (81) is,

$$Z \approx s(L + L_e) \tag{82}$$

and the board looks like an inductor. In this regime, we cannot do any better than the bare board inductive line sL; in order to get close to the bare board line we must have enough decoupling capacitors so that,

$$L_e << L \; . \tag{83}$$

As the frequency increases further, we get a pole at,

$$s^2 = -\frac{1}{L_e C} \tag{84}$$

in equation (81). In general, this pole is at a high frequency; in table 2 it is at 465MHz. Equation (81) shows that we also get a zero at the higher frequency,

$$s^2 = -\frac{1}{L_e C} - \frac{1}{LC} \; . \tag{85}$$

Above this zero, the decoupling capacitors cease to have any effect and the impedance is the bare board impedance $Z = sL$. The next section illustrates all these qualitative features by an example.

4.3 Example of a square board with several species

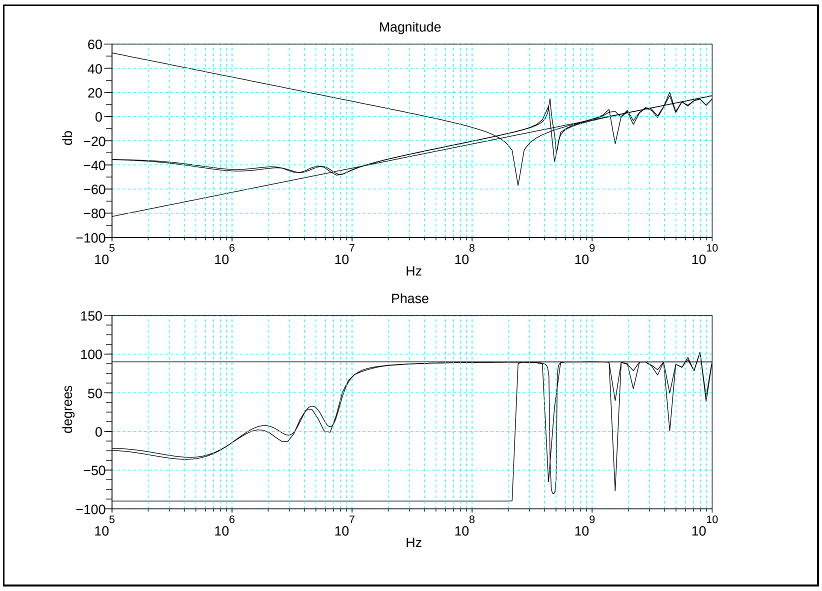

Figure 16: Impedance for a square board with decoupling

parameter	value	meaning
r_0	125.0e-6 m	via-post radius
a	100.0e-3 m	square board size
h	100.0e-6 m	gap between planes
e_r	4.2	relative permittivity of gap
$M^{(1)}$	1	species 1: no. of decoupling caps
$C_d^{(1)}$	470μF	capacitance per device
$L_{esl}^{(1)}$	4.0nH	effective series inductance per device
$R_{esr}^{(1)}$	19mΩ	effective series resistance per device
$M^{(2)}$	3	species 2: no. of decoupling caps
$C_d^{(2)}$	10μF	capacitance per device
$L_{esl}^{(2)}$	1.5nH	effective series inductance per device
$R_{esr}^{(2)}$	20mΩ	effective series resistance per device
$M^{(3)}$	3	species 3: no. of decoupling caps
$C_d^{(3)}$	1μF	capacitance per device
$L_{esl}^{(3)}$	1.5nH	effective series inductance per device
$R_{esr}^{(3)}$	20mΩ	effective series resistance per device
$M^{(4)}$	37	species 4: no. of decoupling caps
$C_d^{(4)}$	100nF	capacitance per device
$L_{esl}^{(4)}$	1.5nH	effective series inductance per device
$R_{esr}^{(4)}$	60mΩ	effective series resistance per device
C	3.714nF	bare board capacitance
L	116.5pH	bare board inductance
$1/(2\pi\sqrt{LC})$	242MHz	bare board zero
L_e	34.6pH	effective decoupling inductance
$1/(2\pi\sqrt{L_eC})$	444MHz	pole

Table 3: Parameters for a square board with decoupling

Table 3 lists the parameters of a square board with 4 species of decoupling capacitors. Figure 16 shows the Bode plots for this square board. The figure overlays several plots.

- Bare board inductive line sL.
- Numerical solution for the bare board impedance.
- Numerical solution for the decoupled board.
- Circuit model calculation for the decoupled board.

The numerical solution (program in appendix B) was set up with the following switches.

```
board_shape=rectangular_board;
a=100.0e-3;
b=100.0e-3;
Nx=5;
Ny=5;
calc_impedance=%T;
plot_bare_board_impedance=%T;
plot_power_plane_impedance=%T;
plot_boundary_and_ports=%T;
```

The port positions for the decoupling capacitiors are generated randomly with the `rectangular_board` switch. The driving port is at the centre of the board. The other parameters in table 3 were also set up on switches. The numbers of capacitors in each species are set up in an array `sn`. The parameters of each species are set up by editing other arrays. Here is a shortened version of the arrays in the program,

```
sn=  [1        ,3       ,3       ,0        ,37       ,0        ];
C=   [470.0e-6,10.0e-6,1.0e-6 ,330.0e-9,100.0e-9,33.0e-9 ];
Lesl=[4.0e-9  ,1.5e-9 ,1.5e-9 ,1.5e-9  ,1.5e-9  ,1.5e-9  ];
Resr=[19.0e-3 ,20.0e-3,20.0e-3,40.0e-3 ,60.0e-3 ,130.0e-3];
```

which shows how to set up the program for the species in table 3.

The circuit model calculation uses the program in appendix C. The circuit model needs the bare board capacitance C and inductance L. These values are given in table 3. The bare board capacitance is,

$$C = \frac{\epsilon_0 \epsilon_r A}{h}$$

where A is the board area. The bare board inductance is found from the position of the bare board zero at 242MHz read from the numerical solution in figure 16. The decoupling species are set up in the same way as in the numerical program with the aid of the species number array `sn` and the other arrays.

Figure 16 shows that the circuit model agrees well with the numerical field solution below 1GHz. The figure illustrates the qualitative behaviour described in section 4.2.

- The low frequency regime of the decoupling capacitors.
- The inductive regime $s(L + L_e)$.
- A pole-zero pair starting at $1/\sqrt{L_e C}$.
- The high frequency bare board regime sL.

The following paragraphs comment on these regimes individually.

The highest self resonance is that of the 100nF species at,

$$\frac{1}{2\pi\sqrt{100 \times 10^{-9} \times 1.5 \times 10^{-9}}} = 13\text{MHz}$$

and the impedance curve in figure 16 is seen to sample the bare board inductive line sL at this frequency.

Beyond the highest decoupling resonance, equation (82) shows that the impedance looks like an inductor $L + L_e$. The effective inductance of all the decoupling capacitors in parallel is given in table 3 as 34.6pH which is comfortably less than the bare board inductance 116.5pH: the impedance is almost as good as it can be in this regime.

The high frequency pole-zero pair near 444MHz signals the end of the effect of the decoupling capacitors; beyond this frequency the system looks like a bare board.

A Program to calculate the analytic field solution for the impedance of a circular board

```
//Program /home/stebla/programs/vector-helmholtz.sci to study
//transients on a pcb. The solution is obtained in reference [1].
//References
//[1] "An attempt to estimate the effect of a switch-mode power
//supply on a low-level circuit", 13th February 2007, Works Volume XXI.
```

```
//[2] "Impedance of a circular printed circuit board", 26th February 2007,
//Works Volume XXI.
//[3] "Solution for a circular pcb with distributed inductance and
//capacitance", 13th March 2007, Works Volume XXII.
//History
//22nd Feb 2007: Created
//26th Feb 2007: Solved for a finite pcb by putting the bc that the
//surface current density is zero at r=r1.
//26th Feb 2007: The solution was tidied-up by following the note [2]
//because the solution in [1] was a rather messy.
//13th Mar 2007: Copied original to vector-helmholtz-v2.sci and modified
//it to use a frequency-dependent permittivity in order to test the
//theory in reference [3].
//8th Feb 2008: The solution was reworked from the Helmholtz equation
//for the magnetic field following appendix A of reference [2] because
//we get an expression for the electric field in terms of the port
//current (the previous version was in terms of the port voltage)
//which can then be directly compared with the electric field in the
//numerical meth in method-of-moments-v5.sci and so is an additional
//way of validating the numerical code.
//11th Feb 2008: Incorporated the solution for a large number of
//decoupling caps using the method of the effective permittivity.
//Usage
//-->exec('vector-helmholtz.sci');
//-->bode(fvec,Z0);//Bode plot of impedance
//-->plot2d(r,imag(E));//Plot of E-field versus radius
//-->plot2d(r,real(H));//Plot of H-field versus radius
//Program controls
//Board parameters
er=4.2;//Relative permittivity of board dielectric ?
r0=125.0e-6;//Radius of via-post for the impressed voltage ?
r1=100.0e-3;//Radius of pcb ?
h=150.0e-6;//Separation of the planes ?
bare_board_impedance=%F;//%T bare board, %F to add decoupling caps ?
M=100;//Number of decoupling capacitors ?
Cd=100.0e-9;//Nominal decoupling capacitance per device ?
Lesl=1.5e-9;//Effective series inductance per device ?
Resr=60.0e-3;//Effective series resistance per device ?
//Impedance Bode plot parameters
f0=100.0e3;// Start frequency for impedance calc?
no_of_decades=5;//No of decades f0 to f0*10^no_of_decades ?
```

```
no_of_data_points=1000;//No of data points for impedance calc ?
//Calculation of E-field versus radius parameters
f=1000.0e6;// Fixed frequency for E-field calc ?
Ie=1;//Impressed current up via-post (port current) ?
r=r0:0.01*r1:r1;//Radial points to calculate E-field solution ?
//SI units
//Physical constants
c=3.0e8;//Speed of light
e0=1.0e7/(4*%pi*c^2);//Permittivity of vacuum
u0=4*%pi*1.0e-7;//Permeability of vacuum
//PCB parameters
//Effective permittivity from decoupling capacitors
s=poly(0,"s");//Variable for the effective permittivity polynomial
Pden=s^2*Lesl*Cd+s*Resr*Cd+1;//Freq-dependence of effective permittivity
ed0=M*Cd*h/(%pi*r1^2);//Zero-frequency addition to permittivity
//Frequency-dependent permittivity function
function eeff=eff_perm(w,ed0,Pden)
  //Computes the effective permittivity for a single species
  //of decoupling capacitor. ed0 is the zero-frequency addition
  //to the usual permittivity and Pden is the polynomial in the
  //denominator.
  eeff=e0*er+ed0/horner(Pden,%i*w);
endfunction;
// ************ Start of the code ******************
w=2*%pi*f;
if bare_board_impedance then
  zeta0=sqrt(u0/(e0*er));//Free space impedance
  k=sqrt(u0*e0*er)*w;//Wavenumber without decoupling
else
  e_eff=eff_perm(w,ed0,Pden);
  zeta0=sqrt(u0/e_eff);//Effective free space impedance
  k=sqrt(u0*e_eff)*w;//Wavenumber with freq-dep permittivity
end;
H10kr0=besselh(0,1,k*r0);//H1nkr=H^1_n(kr) Hankel function of 1st kind
H20kr0=besselh(0,2,k*r0);//H2nkr=H^2_n(kr) Hankel function of 2nd kind
H11kr1=besselh(1,1,k*r1);
H21kr1=besselh(1,2,k*r1);
H11kr0=besselh(1,1,k*r0);
H21kr0=besselh(1,2,k*r0);
for j=1:length(r);
  H10kr=besselh(0,1,k*r(j));
```

```
  H20kr=besselh(0,2,k*r(j));
  H11kr=besselh(1,1,k*r(j));
  H21kr=besselh(1,2,k*r(j));
  H(j)=(Ie/(2*%pi*r0))*(H11kr*H21kr1-H21kr*H11kr1)/(H11kr0*H21kr1-H21kr0*H11kr1)
  E(j)=-%i*zeta0*(Ie/(2*%pi*r0))*..
             (H10kr*H21kr1-H20kr*H11kr1)/(H11kr0*H21kr1-H21kr0*H11kr1);
end;
n=no_of_data_points;//No of data points for impedance calc ?
m=no_of_decades;//No of decades f0 to f0*10^m
for j=1:n;
  fvec(j)=f0*10^(m*(j-1)/(n-1));
  w=2*%pi*fvec(j);
  if bare_board_impedance then
    zeta0=sqrt(u0/(e0*er));//Free space impedance
    k=sqrt(u0*e0*er)*w;//Wavenumber without decoupling
  else
    e_eff=eff_perm(w,ed0,Pden);
    zeta0=sqrt(u0/e_eff);//Effective free space impedance
    k=sqrt(u0*e_eff)*w;//Wavenumber with freq-dep permittivity
  end;
  H10kr0=besselh(0,1,k*r0);//H1nkr=H^1_n(kr) Hankel function of 1st kind
  H20kr0=besselh(0,2,k*r0);//H2nkr=H^2_n(kr) Hankel function of 2nd kind
  H11kr1=besselh(1,1,k*r1);
  H21kr1=besselh(1,2,k*r1);
  H11kr0=besselh(1,1,k*r0);
  H21kr0=besselh(1,2,k*r0);
  Z0(j)=(1/(2*%pi*%i))*zeta0*(h/r0)*..
        (H20kr0*H11kr1-H10kr0*H21kr1)/(H11kr0*H21kr1-H21kr0*H11kr1);
end;
```

B Program to calculate the numerical field solution for the impedance of a general board

```
//Program /home/stebla/programs/scilab/method-of-moments-v5.sci
//to solve the scalar Helmholtz equation for the E-field on a pcb
//in the presence of decoupling capacitors. Lines ending in ?
//can be editted to change the program parameters.
//
```

```
//References
//[1] "The impedance of a pcb with discrete decoupling capacitors
//calculated using the method of moments of R.F. Harrington", 4th
//March 2007, Works Volume XXII.
//History
//6th Mar 2007: Created.
//7th Mar 2007: Copied the original method-of-moments.sci to
//method-of-moments-v2.sci because the original had a fundamental
//error which calculated the low frequency impedance as inductive,
//which is unphysical. This version solves the new integral equation
//in section 11 of reference [1].
//8th Mar 2007: Copied the method-of-moments-v2.sci to
//method-of-moments-v3.sci because the v2 variant does not have the
//correct behaviour at low k. The v3 variant uses Scilab's canned
//integration routine intg to calculate the matrix elements move exactly,
//in the hope that this will fix the problem at low k - it does!
//8th Mar 2007: Copied the method-of-moments-v3.sci to
//method-of-moments-v4.sci . The v3 variant has the correct behaviour
//at low k, but it achieves this by doing the "boundary field due to
//boundary field" integral using Scilab's canned integration routine
//intg; this is slow, and we don't need this precision at higher k
//when the Hankel function is better-behaved. Consequently, the v4
//variant has a k cut-off; below the cut-off we use intg, above the
//cut-off we use the crude approximation in the v2 variant.
//8th Mar 2007: method-of-moments-v4-inst1.sci adds the ability
//to set up port coords to put several species of capacitors on a
//rectangular pcb.
//4th Feb 2008: Tidied up the code to make one canonical version which
//can be put on cvs and so all the previous versions can be forgotten.
// The new version is method-of-moments-v5.sci .
//Usage
//-->exec('method-of-moments-v5.sci');
//-->plot2d(rvec,image(Ezvec));//Plot calculated E-field versus radius
//clear();
//Program controls
circular_board=1;//Circular board shape - do not edit
rectangular_board=2;//Rectangular board shape - do not edit
arbitrary_board=3;//Arbitrary board shape - do not edit
board_shape=rectangular_board;//Generate vertices automatically for a circular b
if board_shape==circular_board then
  r1=100.0e-3;//radius for a circular board ?
```

```
  N=20;//No of boundary edges ?
elseif board_shape==rectangular_board then
  a=100.0e-3;//length for a rectangular board ?
  b=100.0e-3;//width for a rectangular board ?
  Nx=5;//Segments along x edge for rect brd ?
  Ny=5;//Segments along y edge for rect brd ?
else //
  N=20;//No of boundary vertices for an arbitrary board ?
end;
er=4.2;//Relative permittivity of board dielectric ?
r0=125.0e-6;//Radius of the via-posts ?
h=100.0e-6;//Separation of the planes ?
plot_boundary_and_ports=%T;// ?
board_display=[-0.1,-0.1,0.1,0.1];//brd disp rect [xmin,ymin,xmax,ymax] ?
calc_field=%F;// ?
f=1000.0e6;// Fixed frequency for E-field calc ?
rvec=r0:0.01*r1:r1;//Radial points to calculate E-field solution ?
calc_impedance=%T;// ?
f0=100.0e3;// Start frequency for impedance calc?
no_of_decades=5;//No of decades f0 to f0*10^no_of_decades ?
no_of_data_points=100;//No of data points for impedance calc ?
plot_bare_board_impedance=%T;// ?
plot_power_plane_impedance=%T;// ?
fcutoff=50.0e9;//Use intg below this frequency ?
// ************** Decoupling caps ? **************
//These arrays specify the decoupling caps by species. The array sn
//is the number of caps in each species on the board. C is the cap
//value for each species, Lesl is the effective series inductance
//for each species and Resr is the effective series resistance for
//each species. These values are just used in the argument list of
//the function standard_cap to calculate the discrete impedance Zd.
//If the model for the TDK MLCC caps is required, the if statement
//in the code has to be modified to call the mlcc function instead
//of standard_cap, because the MLCC model uses extra parameters.
sn=  [1        ,3       ,3       ,0        ,37       ,0        ,0        ];
C=   [470.0e-6,10.0e-6,1.0e-6 ,330.0e-9,100.0e-9,33.0e-9 ,10.0e-9 ];
Lesl=[4.0e-9  ,1.5e-9 ,1.5e-9 ,1.5e-9  ,1.5e-9  ,1.5e-9  ,1.5e-9  ];
Resr=[19.0e-3 ,20.0e-3,20.0e-3,40.0e-3 ,60.0e-3 ,130.0e-3,200.0e-3];
//**********************************************
function v=vertices_for_circular_board(N,r1)
  //Returns the N vertices for a circular board of radius r1.
```

```
  v=zeros(N,2);//Boundary vertices
  for i=1:N,
    phi=(i-1)*2*%pi/N;
    v(i,:)=[r1*cos(phi),r1*sin(phi)];
  end;
endfunction;
function [v,N]=vertices_for_rectangular_board(Nx,Ny,a,b)
  //Returns the N vertices for a rectangular board of size a x b
  //with Nx segments along x and Ny segments along y.
  N=2*(Nx+Ny);
  da=a/Nx;
  db=b/Ny;
  v=zeros(N,2);//Boundary vertices
  for i=1:Nx,
    v(i,:)=[(i-1)*da-0.5*a,-0.5*b];
  end;
  for j=1:Ny,
    v(Nx+j,:)=[0.5*a,(j-1)*db-0.5*b];
  end;
  for i=1:Nx,
    v(Nx+Ny+i,:)=[0.5*a-(i-1)*da,0.5*b];
  end;
  for j=1:Nx,
    v(2*Nx+Ny+j,:)=[-0.5*a,0.5*b-(j-1)*db];
  end;
endfunction;

function pe=random_ports_for_circular_board(M,r1)
  //A function to automatically generate M ports uniformly
  //over a circular board of radius r1.
  pe=zeros(M,2);
  pe(1,:)=1.0e-3*[0,0];//Driver port coords at centre
  for i=2:M,
    u=rand();//Two random number on [0,1]
    v=rand();
    r=r1*sqrt(u);
    theta=2*%pi*v;
    pe(i,:)=[r*cos(theta),r*sin(theta)];
  end;
endfunction;
function pe=random_ports_for_rectangular_board(M,a,b)
```

```
  //A function to automatically generate M ports uniformly
  //over a rectangular board of dimensions a x b.
  pe=zeros(M,2);
  pe(1,:)=1.0e-3*[0,0];//Driver port coords at centre
  for i=2:M,
    u=rand();//Two random number on [0,1]
    v=rand();
    x=a*(u-0.5);
    y=b*(v-0.5);
    pe(i,:)=[x,y];
  end;
endfunction;
function pe=hand_crafted_ports(M)
  // A function to encapsulate the edits for hand-crafted
  //ports. Clearly M must agree with the vertices
  //actually assigned.
  pe=zeros(M,2);
  pe(1,:)=1.0e-3*[0,0];//Driver port coords
  pe(2,:)=1.0e-3*[-15,-30];
  pe(3,:)=1.0e-3*[-15,-35];
  pe(4,:)=1.0e-3*[-10,-40];
  pe(5,:)=1.0e-3*[-10,-30];
  pe(6,:)=1.0e-3*[-10,-35];
  pe(7,:)=1.0e-3*[-5,-40];
  pe(8,:)=1.0e-3*[-5,-35];
  pe(9,:)=1.0e-3*[-5,-30];
endfunction;
function v=hand_crafted_vertices(N)
  // A function to encapsulate the edits for hand-crafted
  //vertices. Clearly N must agree with the vertices
  //actually assigned.
  v=zeros(N,2);//Boundary vertices
  v(1,:)=1.0e-3*[-50,-50];
  v(2,:)=1.0e-3*[-30,-50];
  v(3,:)=1.0e-3*[-10,-50];
  v(4,:)=1.0e-3*[10,-50];
  v(5,:)=1.0e-3*[30,-50];
  v(6,:)=1.0e-3*[50,-50];
  v(7,:)=1.0e-3*[50,-30];
  v(8,:)=1.0e-3*[50,-10];
  v(9,:)=1.0e-3*[50,10];
```

```
  v(10,:)=1.0e-3*[50,30];
  v(11,:)=1.0e-3*[50,50];
  v(12,:)=1.0e-3*[30,50];
  v(13,:)=1.0e-3*[10,50];
  v(14,:)=1.0e-3*[-10,50];
  v(15,:)=1.0e-3*[-30,50];
  v(16,:)=1.0e-3*[-50,50];
  v(17,:)=1.0e-3*[-50,30];
  v(18,:)=1.0e-3*[-50,10];
  v(19,:)=1.0e-3*[-50,-10];
  v(20,:)=1.0e-3*[-50,-30];
endfunction;
function u=direction(p1,p2)
  //unit vector from p2 to p1; u=(p1-p2)/|p1-p2|
  u=p1-p2;
  len=sqrt(u*u');
  u=u/len;
endfunction;
function d=distance(p1,p2)
  //distance from p2 to p1; d=|p1-p2|
  u=p1-p2;
  d=sqrt(u*u');
endfunction;
function D1_kernel=D1_int(t,k,p,dl,e,p0)
  //Function for integral over a boundary segment.
  //The midpoint of the segment is p, the segment edge
  //vector is dl, the outward unit normal to the segment
  //is e, the field point is p0. This function must not
  //be used when the field point p0 taken on the segment
  //itself, that case has to be done analytically because
  //of the singularity in the Hankel function. The path is
  //parameterised by t which goes from -1/2 to 1/2. The
  //wavenumber is k. Scilab's intg canned integration
  //routine only seems to do real integrals, so we have to
  //split the integral up into real and imaginary parts.
  //This does the real part.
  pt=p+t*dl;
  d=distance(pt,p0);
  u=direction(pt,p0);
  dircos=e*u';
  H21kd=besselh(1,2,k*d);
```

```
  D1_kernel=real(H21kd*dircos);
endfunction;
function D2_kernel=D2_int(t,k,p,dl,e,p0)
  //Function for integral over a boundary segment.
  //The midpoint of the segment is p, the segment edge
  //vector is dl, the outward unit normal to the segment
  //is e, the field point is p0. This function must not
  //be used when the field point p0 taken on the segment
  //itself, that case has to be done analytically because
  //of the singularity in the Hankel function. The path is
  //parameterised by t which goes from -1/2 to 1/2. The
  //wavenumber is k. Scilab's intg canned integration
  //routine only seems to do real integrals, so we have to
  //split the integral up into real and imaginary parts.
  //This does the imaginary part.
  pt=p+t*dl;
  d=distance(pt,p0);
  u=direction(pt,p0);
  dircos=e*u';
  H21kd=besselh(1,2,k*d);
  D2_kernel=imag(H21kd*dircos);
endfunction;
function [CNxM,DNxN,CMxM,DMxN]=port_matrices(k)
  //Calculates the port matrices for a given wavenumber k
  //Assumes that the geometry has been set up in the
  //arrays p,pe,e,dl.
  ea=1.0e-10;//abolute error for intg
  er=1.0e-3;//realtive error for intg
  //Construct the CNxM matrix (boundary fields due to port currents)
  CNxM=zeros(N,M);
  for i=1:N,
    for l=1:M,
      H20kd=besselh(0,2,k*distance(p(i,:),pe(l,:)));
      CNxM(i,l)=-0.25*k*zeta0*H20kd;
    end;
  end;
  //Construct the DNxN matrix (boundary fields due to boundary fields)
  DNxN=zeros(N,N);
  for i=1:N,
    for j=1:N,
      if i==j then
```

```
      DNxN(j,j)=0.5;
    else
      Dl=sqrt(dl(j,:)*dl(j,:)');//Segment length
      if k<kcutoff then
        int1=intg(-0.5,0.5,list(D1_int,k,p(j,:),dl(j,:),e(j,:),p(i,:)),ea,er);
        int2=intg(-0.5,0.5,list(D2_int,k,p(j,:),dl(j,:),e(j,:),p(i,:)),ea,er);
        DNxN(i,j)=-0.25*%i*k*Dl*(int1+%i*int2);
      else
        d=distance(p(i,:),p(j,:));
        u=direction(p(i,:),p(j,:));
        dircos=e(j,:)*u';
        H21kd=besselh(1,2,k*d);
        DNxN(i,j)=0.25*%i*k*Dl*H21kd*dircos;
      end;
    end;
  end;
end;
//Construct the CMxM matrix (port fields due to port currents)
CMxM=zeros(M,M);
//Start by filling in the diagonal
for m=1:M,
    CMxM(m,m)=-0.25*k*zeta0*(1-(2*%i/%pi)*(log(k*r0/2)+gEM-0.5));
end;
for m=1:M-1,
  for l=m+1:M,
    H20kd=besselh(0,2,k*distance(pe(l,:),pe(m,:)));
    CMxM(l,m)=-0.25*k*zeta0*H20kd;
    CMxM(m,l)=CMxM(l,m);//Fill in the symmetric matrix
  end;
end;
//Construct the DMxN matrix (port fields due to boundary fields)
DMxN=zeros(M,N);
for m=1:M,
  for j=1:N,
    Dl=sqrt(dl(j,:)*dl(j,:)');//Segment length
    u=direction(pe(m,:),p(j,:));
    dircos=e(j,:)*u';
    H21kd=besselh(1,2,k*distance(pe(m,:),p(j,:)));
    DMxN(m,j)=0.25*%i*k*Dl*H21kd*dircos;
  end;
end;
```

```
endfunction;
function Ez=E_field(k,EzNx1,IeMx1,p0)
  //Calculates the interior E-field at a point p0
  //given the wavenumber k, the boundary fields EzNx1,
  //and the exciter currents IeMx1. The boundary fields
  //EzNx1 are found by getting the CNxM and DNxN matrices (at the
  //given k),
  //[CNxM,DNxN,CMxM,DMxN]=port_matrices(k);//Wasteful, don't use last pair
  //and then calculating the boundary fields EzNx1 as,
  //EzNx1=inv(eye(N,N)-DNxN)*CNxM*IeMx1;
  //
  //Solution for the E-field in the interior
  Ez=0;
  //Contribution from the port currents
  for l=1:M,
    d=distance(pe(l,:),p0);
    //If the field point p0 is on a port, we have to use the
    //formula worked out in connection with the CMxM matrix that
    //integrates through the singularity.
    if d < r0 then
      Ez=Ez-0.25*k*zeta0*(1-(2*%i/%pi)*(log(k*r0/2)+gEM-0.5))*IeMx1(l);
    else
      H20kd=besselh(0,2,k*d);
      Ez=Ez-0.25*k*zeta0*H20kd*IeMx1(l);
    end;
  end;
  //Contribution from the boundary currents
  //Note, we don't deal with the singularity when we get close to
  //the boundary
  for j=1:N,
    Dl=sqrt(dl(j,:)*dl(j,:)');//Segment length
    d=distance(p(j,:),p0);
    u=direction(p0,p(j,:));
    dircos=e(j,:)*u';
    H21kd=besselh(1,2,k*d);
    Ez=Ez+0.25*%i*k*Dl*H21kd*dircos*EzNx1(j);
  end;
endfunction;
function Z=standard_cap(w,C,Lesl,Resr)
  //Equivalent for a standard cap
  Z=Resr+%i*w*Lesl+1/(%i*w*C);
```

```
endfunction;
function Z=mlcc(w,Cp,Lesl,Re,R,Ri)
  //Equivalent circuit for a TDK MLCC (Multilayer
  //Ceramic Capacitor) from http://www.component.tdk.com
  //w=2*%pi*f
  //Cp=capacitance
  //Lesl=equivalent series inductance
  //Re=electrode resistance
  //R=value in Ohm.Hz from TDK tech notes
  //Ri=insulation resistanceRh=2*%pi*R/w;//Hysteris-derived resistance
  Z=Re+%i*w*Lesl+1/(1/Ri+1/Rh+%i*w*Cp);
endfunction;
//************ Useful constants ****************
//SI units
c=3.0e8;//Speed of light
e0=1.0e7/(4*%pi*c^2);//Permittivity of vacuum
u0=4*%pi*1.0e-7;//Permeability of vacuum
zeta0=sqrt(u0/(e0*er));//Free space impedance
gEM=-dlgamma(1);//Euler-Mascheroni constant (0.5772)
//Cut-off frequency. The integrals which compute the boundary
//E-field due to a boundary E-field need to be done by the intg
//canned routine at low k. The criterion for "low k" is when
//kd ~ 1 where d is the board length because then all effects
//are computed in the regime where the Hankel function changes
//rapidly towards its singularity. Above the cut-off the Hankel
//function is better behaved and we use a crude approximation to
//the integrals which reduces the time for the calculations.
wcutoff=2*%pi*fcutoff;
kcutoff=sqrt(u0*e0*er)*wcutoff;//Wavenumber
// ************ Start of the code ******************
if board_shape==circular_board then
  v=vertices_for_circular_board(N,r1);
elseif board_shape==rectangular_board then
  [v,N]=vertices_for_rectangular_board(Nx,Ny,a,b);
elseif board_shape==arbitrary_board then
  v=hand_crafted_vertices(N);// ?
else
  error('No board specified');
end;
M=1;//Plus 1 for driving port
for s=1:length(sn),
```

```
  M=M+sn(s);//Get number of ports
end;
if board_shape==circular_board then
  pe=random_ports_for_circular_board(M,r1);
elseif board_shape==rectangular_board then
  pe=random_ports_for_rectangular_board(M,a,b);
elseif board_shape==arbitrary_board then
  pe=hand_crafted_ports(M);// ?
else
  error('No board specified');
end;

species=zeros(M,1);//Label each port with the species.
ic=1;//first port is driver port
species(ic)=0;//Use 0 for the driver "species"
for s=1:length(sn),
  for sic=1:sn(s),
    ic=ic+1;//Increment port
    species(ic)=s;//This ic is species s
  end;
end;
//
//Boundary points p, edges dl, and outward unit normals e; this
//section generates automatically from the vertices v
p=zeros(N,2);
dl=zeros(N,2);
e=zeros(N,2);
for i=1:N,
  if i==N then
    p(i,:)=(v(i,:)+v(1,:))/2;
    dl(i,:)=v(1,:)-v(i,:);
  else
    p(i,:)=(v(i,:)+v(i+1,:))/2;
    dl(i,:)=v(i+1,:)-v(i,:);
  end;
  e(i,:)=[dl(i,2),-dl(i,1)]/sqrt(dl(i,:)*dl(i,:)');
end;
if plot_boundary_and_ports then
  //Plot the shape of the pcb
  xbasc();
  printf('Plotting boundary and ports\n');
```

```
  printf('Type return to continue\n');
  for i=1:N,
    xv(1)=p(i,1)-dl(i,1)/2;
    xv(2)=p(i,1)+dl(i,1)/2;
    yv(1)=p(i,2)-dl(i,2)/2;
    yv(2)=p(i,2)+dl(i,2)/2;
    plot2d4(xv,yv,rect=board_display,frameflag=3,style=1);
  end;
  plot2d(pe(:,1)',pe(:,2)',style=-3,rect=board_display);//Plot ports
  pause;
  xbasc();//Clear plot of board ready for bode plot
end;
//E-field at a fixed frequency f
if calc_field then
  Ie=zeros(M,1);
  Ie(1)=1;//Port currents
  w=2*%pi*f;
  k=sqrt(u0*e0*er)*w;//Wavenumber
  [CkNxM,DkNxN,CkMxM,DkMxN]=port_matrices(k);//Wasteful, don't use last pair
  EzNx1=inv(eye(N,N)-DkNxN)*CkNxM*Ie;//Solve for the boundary fields
  phi=0;
  for j=1:length(rvec),
    p0=[rvec(j)*cos(phi),rvec(j)*sin(phi)];
    Ezvec(j)=E_field(k,EzNx1,Ie,p0);
  end;
end;
if calc_impedance then
  //Matrix solution for an M-port system of discrete components
  //connected to the board. This section needs to be hand-crafted
  //to set up the particular mix of capacitors required.
  Zd=zeros(M,M);//The diagonal matrix of discrete impedances
  Zc=zeros(M,M);//The symmetric impedance matrix from the field solution
  Id=zeros(M,1);//Row vector of currents into each port.
  Vd=zeros(M,1);//Port voltages
  ed=zeros(M,1);//Row vector of discrete exciter emfs
  //Exciter emfs
  ed(1)=1;
  n=no_of_data_points;//No of data points for impedance calc
  fvec=zeros(1,n);//Frequency vector (for Bode plot)
  Zvec=zeros(1,n);//Impedance looking into port 1
  Z11vec=zeros(1,n);//Impedance looking into port 1 for bare board
```

```
  m=no_of_decades;//No of decades f0 to f0*10^m
  for j=1:n,//Loop on frequency
    fvec(j)=f0*10^(m*(j-1)/(n-1));
    w=2*%pi*fvec(j);
    k=sqrt(u0*e0*er)*w;//Wavenumber
    [CkNxM,DkNxN,CkMxM,DkMxN]=port_matrices(k);//Using all matrices now
    Zc=-h*(CkMxM+DkMxN*inv(eye(N,N)-DkNxN)*CkNxM);//Impedance matrix
    Z11vec(j)=Zc(1,1);//Bare board impedance into port 1 is Zc(1,1)
    //Calculate the impedance matrix of the discretes
    Zd(1,1)=0;//Port 1 is the exciter port
    for i=2:M,
      is=species(i);//Get the species of cap at the ith port
      //Modify this if statement to explicitly call other cap models ?
      if is<=9 then
        Zd(i,i)=standard_cap(w,C(is),Lesl(is),Resr(is));
      else
        error('This should not happen');
      end
    end;
    //Solve for the currents of the M-port system
    Id=inv(Zd+Zc)*ed;
    Vd=Zc*Id;//Get the port voltages
    Zvec(j)=Vd(1)/Id(1);//Impedance at port 1 for each frequency
    printf('%s%f%s%+f%+fi\n','Zvec(',fvec(j),')=',real(Zvec(j)),imag(Zvec(j)));
  end;
  if plot_power_plane_impedance then
    bode(fvec,Zvec);
  end;
  if plot_bare_board_impedance then
    bode(fvec,Z11vec);
  end;
end;
```

C Program to calculate the impedance of a general board using the circuit model

```
//Program power_plane_circuit_model.sci
//Circuit model for power plane impedance with decoupling.
//Lines ending in ? can be editted to change the program parameters.
```

```
//
f0=100.0e3;// Initial frequency for Bode plot ?
f1=9.99e9;// Final frequency for Bode plot ?
//Bare board inductance and capacitance must be supplied from a
//field solution.
L=116.5e-12;// Bare board inductance ?
C=3.714e-9;// Bare board capacitance ?
// ************** Decoupling caps ? **************
//These arrays specify the decoupling caps by species. The array sn
//is the number of caps in each species on the board. Cd is the cap
//value for each species, Lesl is the effective series inductance
//for each species and Resr is the effective series resistance for
//each species. These values are just used in the argument list of
//the function standard_cap to calculate the discrete impedance Zd.
sn=  [1        ,100    ,100    ,100     ,100     ,100     ,100     ];
Cd=  [470.0e-6,10.0e-6,1.0e-6 ,330.0e-9,100.0e-9,33.0e-9 ,10.0e-9 ];
Lesl=[4.0e-9  ,1.5e-9 ,1.5e-9 ,1.5e-9  ,1.5e-9  ,1.5e-9  ,1.5e-9  ];
Resr=[19.0e-3 ,20.0e-3,20.0e-3,40.0e-3 ,60.0e-3 ,130.0e-3,200.0e-3];
//************************************************
//
// ************ Start of the code ******************
s=poly(0,"s");//Laplace transform variable
Y=s*C;//Y will be the total admittance
for j=1:length(sn),
  if sn(j)>0 then
    Zj=(s*Lesl(j)+1/(s*Cd(j))+Resr(j))/sn(j);//Impedance of jth species
    Yj=1/Zj;//Admittance of jth species
    Y=Y+Yj;
  end;
end;
Z=s*L+1/Y;
s1=syslin('c',Z);
bode(s1,f0,f1,0.01);
```

References

[1] Panofsky and Phillips. *Classical Electricity and Magnetism.* Addison Wesley, 2nd edition, 1969.

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[3] Campbell, Chancelier and Nikoukhah. *Modeling and Simulation in Scilab/Scicos*. Springer 2006.